

Parametrized Trigonometric derived L_p degree of approximation by various smooth singular integral operators

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Abstract

In this work we continue with the study of smooth Gauss-Weierstrass, Poisson-Cauchy and Trigonometric singular integral operators that started in [3], see there chapters 10-14. This time the foundation of our research is a trigonometric Taylor's formula. We prove the parametrized univariate L_p convergence of our operators to the unit operator with rates via Jackson type parametrized inequalities involving the first L_p modulus of continuity. Of interest here is a residual appearing term. Note that our operators are not in general positive.

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1 Introduction

We are motivated by [2], [3] chapters 10-14, and [4], [1]. We use a trigonometric new Taylor formula from [4], see also [1]. Here we consider some very general operators, the smooth Gauss-Weierstrass, Poisson-Cauchy and trigonometric singular integral operators over the real line and we study further their L_p , $p \geq 1$, parametrized convergence properties quantitatively. We establish related parametrized inequalities involving the first L_p , $p \geq 1$, modulus of continuity with respect to L_p , $p \geq 1$, norm. We provide detailed proofs.

Other important motivating articles on the topic are [6]-[10].

For the history of the topic we mention about the monograph [5] of 2012, which was the first complete source to deal exclusively with the classic theory of the approximation of singular integrals to the identity-unit operator. The authors there studied quantitatively the basic approximation properties of the general Picard, Gauss-Weierstrass and Poisson-Cauchy singular integral operators over the real line, which are not positive linear operators. In particular they studied the rate of convergence of these operators to the unit operator, as well as the related simultaneous approximation. This is given via inequalities and with the use of higher order modulus of smoothness of the high order derivative of the involved function. Some of these inequalities are proven to be sharp. Also, they studied the global smoothness preservation property of these operators. Furthermore they gave asymptotic expansions of Voronovskaya type for the error of approximation. They continued with the study of related properties of the general fractional Gauss-Weierstrass and Poisson-Cauchy singular integral operators. These properties were studied with respect to L_p norm, $1 \leq p \leq \infty$. The case of Lipschitz type functions approximation was studied separately and in detail. Furthermode they presented the corresponding general approximation theory of general singular integral operators with lots of applications to, the under focused till then, trigonometric singular integral.

2 Part I: on Gauss-Weierstrass smooth singular integrals

By [1], [4], for $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$, and $f \in C^4(\mathbb{R})$, $a, x \in \mathbb{R}$, we have the following general trigonometric Taylor formula:

$$\begin{aligned}
f(x) - f(a) &= f'(a) \left(\frac{\beta^3 \sin(\alpha(x-a)) - \alpha^3 \sin(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \right) + \\
&f''(a) \left(\frac{\cos(\alpha(x-a)) - \cos(\beta(x-a))}{\beta^2 - \alpha^2} \right) + \\
&f'''(a) \left(\frac{\beta \sin(\alpha(x-a)) - \alpha \sin(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)} \right) + \tag{1} \\
&\frac{2(f''''(a) + (\alpha^2 + \beta^2)f''(a))}{(\alpha\beta)^2(\beta^2 - \alpha^2)} \left(\beta^2 \sin^2\left(\frac{\alpha(x-a)}{2}\right) - \alpha^2 \sin^2\left(\frac{\beta(x-a)}{2}\right) \right) + \\
&\frac{1}{\alpha\beta(\beta^2 - \alpha^2)} \int_a^x [(f''''(t) + (\alpha^2 + \beta^2)f''(t) + \alpha^2\beta^2 f(t)) \\
&\quad - (f''''(a) + (\alpha^2 + \beta^2)f''(a) + \alpha^2\beta^2 f(a))] \\
&\quad [\beta \sin(\alpha(x-t)) - \alpha \sin(\beta(x-t))] dt.
\end{aligned}$$

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, we set

$$\alpha_j := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 1, \dots, r, \\ 1 - \sum_{j=1}^r (-1)^{r-j} \binom{r}{j} j^{-n}, & n = 0, \end{cases} \quad (2)$$

that is

$$\sum_{j=0}^r \alpha_j = 1. \quad (3)$$

Here we consider all $f, f', f'', f''', f^{(4)} \in L_p(\mathbb{R}) \cap C(\mathbb{R})$, $1 \leq p < \infty$.

For $x \in \mathbb{R}$, $\xi > 0$ we consider the Lebesgue integrals, so called smooth Gauss-Weierstrass operators

$$W_{r,\xi}(f, x) = \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} \left(\sum_{j=0}^r \alpha_j f(x + jt) \right) e^{-\frac{t^2}{\xi}} dt, \quad (4)$$

see [5], $W_{r,\xi}$ are not in general positive operators, see [5].

We notice by

$$\frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{\xi}} dt = 1, \quad (5)$$

that

$$W_{r,\xi}(c, x) = c, \text{ where } c \text{ is a constant,} \quad (6)$$

and

$$W_{r,\xi}(f, x) - f(x) = \frac{1}{\sqrt{\pi\xi}} \left(\sum_{j=0}^r \alpha_j \int_{-\infty}^{\infty} (f(x + jt) - f(x)) e^{-\frac{t^2}{\xi}} dt \right). \quad (7)$$

Denote by

$$\omega_1(f, h)_p := \sup_{\substack{t \in \mathbb{R} \\ |t| \leq h}} \|f(x + t) - f(x)\|_{p,x}, \quad (8)$$

the first L_p modulus of smoothness of f , $1 \leq p < \infty$.

By (1) we get that

$$\begin{aligned} f(x + jt) - f(x) &= f'(x) \left(\frac{\beta^3 \sin(\alpha jt) - \alpha^3 \sin(\beta jt)}{\alpha\beta(\beta^2 - \alpha^2)} \right) + \\ &f''(x) \left(\frac{\cos(\alpha jt) - \cos(\beta jt)}{\beta^2 - \alpha^2} \right) + \\ &f'''(x) \left(\frac{\beta \sin(\alpha jt) - \alpha \sin(\beta jt)}{\alpha\beta(\beta^2 - \alpha^2)} \right) + \end{aligned}$$

$$\begin{aligned}
& \frac{2(f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x))}{(\alpha\beta)^2(\beta^2 - \alpha^2)} \left(\beta^2 \sin^2\left(\frac{\alpha jt}{2}\right) - \alpha^2 \sin^2\left(\frac{\beta jt}{2}\right) \right) + \\
& \frac{1}{\alpha\beta(\beta^2 - \alpha^2)} \int_x^{x+jt} \left[(f^{(4)}(s) + (\alpha^2 + \beta^2)f''(s) + \alpha^2\beta^2 f(s)) \right. \\
& \quad \left. - (f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x) + \alpha^2\beta^2 f(x)) \right] \\
& \quad [\beta \sin(\alpha(x+jt-s)) - \alpha \sin(\beta(x+jt-s))] ds,
\end{aligned} \tag{9}$$

or better,

$$\begin{aligned}
f(x+jt) - f(x) &= f'(x) \left(\frac{\beta^3 \sin(\alpha jt) - \alpha^3 \sin(\beta jt)}{\alpha\beta(\beta^2 - \alpha^2)} \right) + \\
& \quad f''(x) \left(\frac{\cos(\alpha jt) - \cos(\beta jt)}{\beta^2 - \alpha^2} \right) + \\
& \quad f'''(x) \left(\frac{\beta \sin(\alpha jt) - \alpha \sin(\beta jt)}{\alpha\beta(\beta^2 - \alpha^2)} \right) + \\
& \quad \frac{2(f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x))}{(\alpha\beta)^2(\beta^2 - \alpha^2)} \left(\beta^2 \sin^2\left(\frac{\alpha jt}{2}\right) - \alpha^2 \sin^2\left(\frac{\beta jt}{2}\right) \right) + \\
& \quad \frac{1}{\alpha\beta(\beta^2 - \alpha^2)} \int_0^{jt} \left[(f^{(4)}(x+z) + (\alpha^2 + \beta^2)f''(x+z) + \alpha^2\beta^2 f(x+z)) \right. \\
& \quad \left. - (f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x) + \alpha^2\beta^2 f(x)) \right] \\
& \quad [\beta \sin(\alpha(jt-z)) - \alpha \sin(\beta(jt-z))] dz.
\end{aligned} \tag{10}$$

Furthermore it holds

$$\begin{aligned}
& \sum_{j=0}^r \alpha_j [f(x+jt) - f(x)] = \\
& \left(\frac{f'(x)}{\alpha\beta(\beta^2 - \alpha^2)} \right) \left[\beta^3 \sum_{j=0}^r \alpha_j \sin(\alpha jt) - \alpha^3 \sum_{j=0}^r \alpha_j \sin(\beta jt) \right] + \\
& \left(\frac{f''(x)}{\beta^2 - \alpha^2} \right) \left[\sum_{j=0}^r \alpha_j \cos(\alpha jt) - \sum_{j=0}^r \alpha_j \cos(\beta jt) \right] + \\
& \left(\frac{f^{(3)}(x)}{\alpha\beta(\beta^2 - \alpha^2)} \right) \left[\beta \sum_{j=0}^r \alpha_j \sin(\alpha jt) - \alpha \sum_{j=0}^r \alpha_j \sin(\beta jt) \right] +
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{2(f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x))}{(\alpha\beta)^2(\beta^2 - \alpha^2)} \right) \left(\beta^2 \sum_{j=0}^r \alpha_j \sin^2 \left(\frac{\alpha jt}{2} \right) - \alpha^2 \sum_{j=0}^r \alpha_j \sin^2 \left(\frac{\beta jt}{2} \right) \right) + \\
& \frac{1}{\alpha\beta(\beta^2 - \alpha^2)} \sum_{j=0}^r \alpha_j \int_0^{jt} \left[(f^{(4)}(x+z) + (\alpha^2 + \beta^2)f''(x+z) + \alpha^2\beta^2 f(x+z)) \right. \\
& \quad \left. - (f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x) + \alpha^2\beta^2 f(x)) \right] \\
& \quad [\beta \sin(\alpha(jt-z)) - \alpha \sin(\beta(jt-z))] dz,
\end{aligned} \tag{11}$$

or better

$$\begin{aligned}
& \sum_{j=0}^r \alpha_j [f(x+jt) - f(x)] = \\
& \left(\frac{f'(x)}{\alpha\beta(\beta^2 - \alpha^2)} \right) \left[\beta^3 \sum_{j=0}^r \alpha_j \sin(\alpha jt) - \alpha^3 \sum_{j=0}^r \alpha_j \sin(\beta jt) \right] + \\
& \left(\frac{f''(x)}{\beta^2 - \alpha^2} \right) \left[\sum_{j=0}^r \alpha_j \cos(\alpha jt) - \sum_{j=0}^r \alpha_j \cos(\beta jt) \right] + \\
& \left(\frac{f^{(3)}(x)}{\alpha\beta(\beta^2 - \alpha^2)} \right) \left[\beta \sum_{j=0}^r \alpha_j \sin(\alpha jt) - \alpha \sum_{j=0}^r \alpha_j \sin(\beta jt) \right] + \\
& \left(\frac{2(f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x))}{(\alpha\beta)^2(\beta^2 - \alpha^2)} \right) \left(\beta^2 \sum_{j=0}^r \alpha_j \sin^2 \left(\frac{\alpha jt}{2} \right) - \alpha^2 \sum_{j=0}^r \alpha_j \sin^2 \left(\frac{\beta jt}{2} \right) \right) + \\
& \frac{1}{\alpha\beta(\beta^2 - \alpha^2)} \sum_{j=0}^r \alpha_j j \int_0^t \left[(f^{(4)}(x+jw) + (\alpha^2 + \beta^2)f''(x+jw) + \alpha^2\beta^2 f(x+jw)) \right. \\
& \quad \left. - (f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x) + \alpha^2\beta^2 f(x)) \right] \\
& \quad [\beta \sin(\alpha j(t-w)) - \alpha \sin(\beta j(t-w))] dw.
\end{aligned} \tag{12}$$

We call

$$\begin{aligned}
R & := R(t) := \frac{1}{\alpha\beta(\beta^2 - \alpha^2)} \sum_{j=0}^r \alpha_j j \\
& \int_0^t \left[(f^{(4)}(x+jw) + (\alpha^2 + \beta^2)f''(x+jw) + \alpha^2\beta^2 f(x+jw)) \right. \\
& \quad \left. - (f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x) + \alpha^2\beta^2 f(x)) \right] \\
& \quad [\beta \sin(\alpha j(t-w)) - \alpha \sin(\beta j(t-w))] dw, \quad \forall t \in \mathbb{R}.
\end{aligned} \tag{13}$$

We set

$$\begin{aligned}
E_1(x) &:= W_{r,\xi}(f, x) - f(x) - \left(\frac{f'(x)}{\alpha\beta(\beta^2 - \alpha^2)} \right) \\
&\left[\beta^3 \sum_{j=0}^r \alpha_j \frac{1}{\sqrt{\pi\xi}} \left(\int_{-\infty}^{\infty} \sin(\alpha jt) e^{-\frac{t^2}{\xi}} dt \right) - \alpha^3 \sum_{j=0}^r \alpha_j \frac{1}{\sqrt{\pi\xi}} \left(\int_{-\infty}^{\infty} \sin(\beta jt) e^{-\frac{t^2}{\xi}} dt \right) \right] - \\
&\quad \left(\frac{f''(x)}{\beta^2 - \alpha^2} \right) \\
&\left[\sum_{j=0}^r \alpha_j \frac{1}{\sqrt{\pi\xi}} \left(\int_{-\infty}^{\infty} \cos(\alpha jt) e^{-\frac{t^2}{\xi}} dt \right) - \sum_{j=0}^r \alpha_j \frac{1}{\sqrt{\pi\xi}} \left(\int_{-\infty}^{\infty} \cos(\beta jt) e^{-\frac{t^2}{\xi}} dt \right) \right] - \\
&\quad \left(\frac{f^{(3)}(x)}{\alpha\beta(\beta^2 - \alpha^2)} \right) \\
&\left[\beta \sum_{j=0}^r \alpha_j \frac{1}{\sqrt{\pi\xi}} \left(\int_{-\infty}^{\infty} \sin(\alpha jt) e^{-\frac{t^2}{\xi}} dt \right) - \alpha \sum_{j=0}^r \alpha_j \frac{1}{\sqrt{\pi\xi}} \left(\int_{-\infty}^{\infty} \sin(\beta jt) e^{-\frac{t^2}{\xi}} dt \right) \right] - \\
&\quad (14) \\
&\quad \left(\frac{2(f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x))}{(\alpha\beta)^2(\beta^2 - \alpha^2)} \right) \\
&\left[\beta^2 \sum_{j=0}^r \alpha_j \frac{1}{\sqrt{\pi\xi}} \left(\int_{-\infty}^{\infty} \sin^2\left(\frac{\alpha jt}{2}\right) e^{-\frac{t^2}{\xi}} dt \right) - \alpha^2 \sum_{j=0}^r \alpha_j \frac{1}{\sqrt{\pi\xi}} \left(\int_{-\infty}^{\infty} \sin^2\left(\frac{\beta jt}{2}\right) e^{-\frac{t^2}{\xi}} dt \right) \right] \\
&= \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} R(t) e^{-\frac{t^2}{\xi}} dt.
\end{aligned}$$

Next we simplify $E_1(x)$:

We observe that

$$\int_{-\infty}^{\infty} \sin(\alpha jt) e^{-\frac{t^2}{\xi}} dt = \int_{-\infty}^0 \sin(\alpha jt) e^{-\frac{t^2}{\xi}} dt + \int_0^{\infty} \sin(\alpha jt) e^{-\frac{t^2}{\xi}} dt. \quad (15)$$

Notice $-\infty \leq t \leq 0 \Rightarrow \infty \geq -t \geq 0$. So that

$$\begin{aligned}
&\int_{-\infty}^0 \sin(\alpha jt) e^{-\frac{|t^2|}{\xi}} dt = - \int_{-\infty}^0 \sin(\alpha j(-(-t))) e^{-\frac{t^2}{\xi}} d(-t) = \\
&- \int_{-\infty}^0 (-\sin(\alpha j(-t))) e^{-\frac{t^2}{\xi}} d(-t) = \int_{-\infty}^0 \sin(\alpha j(-t)) e^{-\frac{t^2}{\xi}} d(-t) = \quad (16) \\
&\int_{\infty}^0 \sin(\alpha jt) e^{-\frac{t^2}{\xi}} dt = - \int_0^{\infty} \sin(\alpha jt) e^{-\frac{t^2}{\xi}} dt.
\end{aligned}$$

So that

$$\int_{-\infty}^{\infty} \sin(\alpha jt) e^{-\frac{t^2}{\xi}} dt = 0,$$

and all sine integrals in (14) are zeros.

Furthermore we have that

$$\begin{aligned} \int_{-\infty}^{\infty} \sin^2\left(\frac{\alpha jt}{2}\right) e^{-\frac{t^2}{\xi}} dt &= 2 \int_0^{\infty} \sin^2\left(\frac{\alpha jt}{2}\right) e^{-\frac{t^2}{\xi}} dt \\ &= 2\sqrt{\xi} \int_0^{\infty} \sin^2\left(\left(\frac{\alpha j\sqrt{\xi}}{2}\right) \frac{t}{\sqrt{\xi}}\right) e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2} d\frac{t}{\sqrt{\xi}} \\ (\frac{t}{\sqrt{\xi}} =: x, \text{ and } \frac{\alpha j\sqrt{\xi}}{2} = \beta_1) \\ &= 2\sqrt{\xi} \int_0^{\infty} \sin^2(\beta_1 x) e^{-x^2} dx = 2\sqrt{\xi} \frac{1}{4} \sqrt{\pi} e^{-\beta_1^2} (e^{\beta_1^2} - 1) \\ &= \frac{\sqrt{\pi\xi}}{2} (1 - e^{-\beta_1^2}) = \frac{\sqrt{\pi\xi}}{2} \left(1 - e^{-\frac{\alpha^2 j^2 \xi}{4}}\right). \end{aligned} \quad (17)$$

That is

$$\int_{-\infty}^{\infty} \sin^2\left(\frac{\alpha jt}{2}\right) e^{-\frac{t^2}{\xi}} dt = \frac{\sqrt{\pi\xi}}{2} \left(1 - e^{-\frac{\alpha^2 j^2 \xi}{4}}\right), \quad (18)$$

and similarly,

$$\int_{-\infty}^{\infty} \sin^2\left(\frac{\beta jt}{2}\right) e^{-\frac{t^2}{\xi}} dt = \frac{\sqrt{\pi\xi}}{2} \left(1 - e^{-\frac{\beta^2 j^2 \xi}{4}}\right). \quad (19)$$

Next, we treat

$$\begin{aligned} \int_{-\infty}^{\infty} \cos(\alpha jt) e^{-\frac{t^2}{\xi}} dt &= 2 \int_0^{\infty} \cos(\alpha jt) e^{-\frac{t^2}{\xi}} dt = \\ &= 2\sqrt{\xi} \int_0^{\infty} \cos\left(\left(\frac{\alpha j\sqrt{\xi}}{2}\right) \frac{t}{\sqrt{\xi}}\right) e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2} d\frac{t}{\sqrt{\xi}} \\ (\frac{t}{\sqrt{\xi}} =: x, \text{ and } \frac{\alpha j\sqrt{\xi}}{2} =: \beta_1) \end{aligned} \quad (20)$$

$$= 2\sqrt{\xi} \int_0^{\infty} \cos(\beta_1 x) e^{-x^2} dx = 2\sqrt{\xi} \frac{1}{2} \sqrt{\pi} e^{-\frac{\beta_1^2}{4}} = \sqrt{\pi\xi} e^{-\frac{\alpha^2 j^2 \xi}{16}}.$$

That is

$$\int_{-\infty}^{\infty} \cos(\alpha jt) e^{-\frac{t^2}{\xi}} dt = \sqrt{\pi\xi} e^{-\frac{\alpha^2 j^2 \xi}{16}}, \quad (21)$$

and similarly,

$$\int_{-\infty}^{\infty} \cos(\beta jt) e^{-\frac{t^2}{\xi}} dt = \sqrt{\pi\xi} e^{-\frac{\beta^2 j^2 \xi}{16}}. \quad (22)$$

Hence, we have the simplified expression,

$$\begin{aligned}
E_1(x) &= W_{r,\xi}(f, x) - f(x) - \\
&\left(\frac{f''(x)}{\beta^2 - \alpha^2} \right) \left[\sum_{j=0}^r \alpha_j \left(e^{-\frac{\alpha^2 j^2 \xi}{16}} - e^{-\frac{\beta^2 j^2 \xi}{16}} \right) \right] - \\
&\left(\frac{f^{(4)}(x) + (\alpha^2 + \beta^2) f''(x)}{(\alpha\beta)^2 (\beta^2 - \alpha^2)} \right) \\
&\left[\beta^2 \sum_{j=0}^r \alpha_j \left(1 - e^{-\frac{\alpha^2 j^2 \xi}{4}} \right) - \alpha^2 \sum_{j=0}^r \alpha_j \left(1 - e^{-\frac{\beta^2 j^2 \xi}{4}} \right) \right] = \\
&\frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} R(t) e^{-\frac{t^2}{\xi}} dt. \tag{23}
\end{aligned}$$

It follows a parametrized L_p ($p > 1$) approximation result for $W_{r,\xi}$, $\xi > 0$.

Theorem 1 *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$, and let all $f, f', f'', f^{(3)}, f^{(4)} \in L_p(\mathbb{R}) \cap C(\mathbb{R})$, $1 < p < \infty$. Here $q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; $\xi > 0$, $x \in \mathbb{R}$. Then*

$$\begin{aligned}
\|E_1(x)\|_p &= \|W_{r,\xi}(f, x) - f(x) - \\
&\frac{f''(x)}{(\beta^2 - \alpha^2)} \left[\sum_{j=0}^r \alpha_j \left(e^{-\frac{\alpha^2 j^2 \xi}{16}} - e^{-\frac{\beta^2 j^2 \xi}{16}} \right) \right] - \\
&\left(\frac{f^{(4)}(x) + (\alpha^2 + \beta^2) f''(x)}{(\alpha\beta)^2 (\beta^2 - \alpha^2)} \right) \\
&\left[\beta^2 \sum_{j=0}^r \alpha_j \left(1 - e^{-\frac{\alpha^2 j^2 \xi}{4}} \right) - \alpha^2 \sum_{j=0}^r \alpha_j \left(1 - e^{-\frac{\beta^2 j^2 \xi}{4}} \right) \right] \|_p \leq \tag{24} \\
&\frac{\left(\sum_{j=1}^r |\alpha_j| j^2 \right)^{\frac{1}{q}} 4 \cdot 2^{\frac{1}{p}}}{(\sqrt{\pi})^{\frac{1}{p}} |\beta^2 - \alpha^2| (q+1)^{\frac{1}{q}} (p+1)^{\frac{1}{p}}} \\
&\left[\sum_{j=1}^r |\alpha_j| j^{p+2} \left(\frac{1}{(3p+1)} + \Gamma(3p+1) \right) \right]^{\frac{1}{p}} \\
&\omega_1 \left(f^{(4)} + (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f, \sqrt{\xi} \right)_p \cdot \xi =: \Phi_1(\xi) \rightarrow 0,
\end{aligned}$$

as $\xi \rightarrow 0$.

Above Γ stands for the gamma function.

Proof. Call the function

$$F := f^{(4)} + (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f. \quad (25)$$

Then, we get

$$R = R(t) = \frac{1}{\alpha\beta(\beta^2 - \alpha^2)} \sum_{j=0}^r \alpha_j j \int_0^t [F(x + jw) - F(x)] [\beta \sin(\alpha j(t - w)) - \alpha \sin(\beta j(t - w))] dw, \quad \forall t \in \mathbb{R}. \quad (26)$$

We isolate and study

$$I := \int_0^t [F(x + jw) - F(x)] [\beta \sin(\alpha j(t - w)) - \alpha \sin(\beta j(t - w))] dw, \quad \forall t \in \mathbb{R}. \quad (27)$$

For $t < 0$, we have that

$$|I| = \left| \int_t^0 [F(x + jw) - F(x)] [\beta \sin(\alpha j(t - w)) - \alpha \sin(\beta j(t - w))] dw \right| \leq \int_t^0 |F(x + jw) - F(x)| |\beta \sin(\alpha j(t - w)) - \alpha \sin(\beta j(t - w))| dw \leq \quad (28)$$

$$2|\alpha| |\beta| j \int_t^0 |F(x + jw) - F(x)| (w - t) dw =$$

$$-2|\alpha| |\beta| j \int_t^0 |F(x - j(-w)) - F(x)| (-t - (-w)) d(-w) =$$

$$(t \leq w \leq 0 \Rightarrow -t \geq -w =: \theta \geq 0)$$

$$-2|\alpha| |\beta| j \int_{-t}^0 |F(x - j\theta) - F(x)| (-t - \theta) d\theta =$$

$$2|\alpha| |\beta| j \int_0^{-t} |F(x - j\theta) - F(x)| (-t - \theta) d\theta =$$

$$2|\alpha\beta| j \int_0^{|t|} |F(x + \text{sign}(t)j\theta) - F(x)| (|t| - \theta) d\theta.$$

So, we have proved that

$$|I| \leq 2|\alpha\beta| j \int_0^{|t|} |F(x + \text{sign}(t)j\theta) - F(x)| (|t| - \theta) d\theta, \quad \forall t \in \mathbb{R}, \quad (29)$$

and, by (26),

$$|R(t)| \leq \frac{2}{|\beta^2 - \alpha^2|} \sum_{j=0}^r |\alpha_j| j^2 \int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)| (|t| - \theta) d\theta, \quad (30)$$

$\forall t \in \mathbb{R}$.

By (14), we have

$$E_1(x) = \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} R(t) e^{-\frac{t^2}{\xi}} dt. \quad (31)$$

Hence it holds $(p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1)$,

$$\begin{aligned} \int_{-\infty}^{\infty} |E_1(x)|^p dx &= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{\pi\xi}} \left| \int_{-\infty}^{\infty} R(t) e^{-\frac{t^2}{\xi}} dt \right| \right)^p dx \leq \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)| \frac{e^{-\frac{t^2}{\xi}}}{\sqrt{\pi\xi}} dt \right)^p dx \leq \\ &= \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)|^p e^{-\frac{t^2}{\xi}} dt \right) dx \stackrel{(30)}{\leq} \frac{1}{\sqrt{\pi\xi}} \frac{2^p}{|\beta^2 - \alpha^2|^p} \quad (32) \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left[\sum_{j=1}^r |\alpha_j| j^2 \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)| (|t| - \theta) d\theta \right) \right]^p e^{-\frac{t^2}{\xi}} dt \right) dx \\ &\stackrel{\text{(by Jensen inequality)}}{\leq} \frac{1}{\sqrt{\pi\xi}} \frac{2^p}{|\beta^2 - \alpha^2|^p} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(\sum_{j=1}^r |\alpha_j| j^2 \right)^{p-1} \right. \\ &\quad \left. \left[\sum_{j=1}^r |\alpha_j| j^2 \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)| (|t| - \theta) d\theta \right)^p \right] e^{-\frac{t^2}{\xi}} dt \right) dx \\ &\leq \frac{2^p \left(\sum_{j=1}^r |\alpha_j| j^2 \right)^{p-1}}{\sqrt{\pi\xi} |\beta^2 - \alpha^2|^p} \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \right. \right. \\ &\quad \left. \left. \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)|^p d\theta \right) \left(\int_0^{|t|} (|t| - \theta)^q d\theta \right)^{\frac{p}{q}} e^{-\frac{t^2}{\xi}} dt \right) dx \right] = \\ &= \frac{2^p \left(\sum_{j=1}^r |\alpha_j| j^2 \right)^{p-1}}{\sqrt{\pi\xi} |\beta^2 - \alpha^2|^p (q+1)^{p-1}} \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \right. \right. \quad (33) \end{aligned}$$

$$\begin{aligned}
& \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)|^p d\theta \right) |t|^{(q+1)(p-1)} e^{-\frac{t^2}{\xi}} dt \Big) dx \Big] = \\
& \text{(call } c := \frac{2^p \left(\sum_{j=1}^r |\alpha_j| j^2 \right)^{p-1}}{\sqrt{\pi\xi} |\beta^2 - \alpha^2|^p (q+1)^{p-1}}) \\
& c \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(\int_{-\infty}^{\infty} \right. \right. \right. \\
& \quad \left. \left. \left. |F(x + j \operatorname{sign}(t)\theta) - F(x)|^p dx \right) d\theta \right) |t|^{(q+1)(p-1)} e^{-\frac{t^2}{\xi}} dt \right] \leq \\
& c \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \omega_1 \left(F, \sqrt{\xi} \frac{j\theta}{\sqrt{\xi}} \right)_p^p d\theta \right) |t|^{(q+1)(p-1)} e^{-\frac{t^2}{\xi}} dt \right] \leq \quad (34) \\
& c \omega_1 \left(F, \sqrt{\xi} \right)_p^p \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(1 + \frac{j\theta}{\sqrt{\xi}} \right)^p d\theta \right) |t|^{(q+1)(p-1)} e^{-\frac{t^2}{\xi}} dt \right] = \\
& c \omega_1 \left(F, \sqrt{\xi} \right)_p^p \frac{\sqrt{\xi}}{(p+1)} \left[\sum_{j=1}^r |\alpha_j| j \int_{-\infty}^{\infty} \left[\left(1 + \frac{j}{\sqrt{\xi}} |t| \right)^{p+1} - 1 \right] |t|^{(q+1)(p-1)} e^{-\frac{t^2}{\xi}} dt \right] \leq \\
& \frac{2^p c \sqrt{\xi}}{(p+1)} \omega_1 \left(F, \sqrt{\xi} \right)_p^p \left[\sum_{j=1}^r |\alpha_j| j \frac{j^{p+1}}{(\sqrt{\xi})^{p+1}} \int_{-\infty}^{\infty} |t|^{p+1} |t|^{(q+1)(p-1)} e^{-\frac{t^2}{\xi}} dt \right] = \\
& \frac{2^{p+1} c (\sqrt{\xi})^{-p}}{(p+1)} \omega_1 \left(F, \sqrt{\xi} \right)_p^p \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \int_0^{\infty} t^{(q+1)(p-1) + (p+1)} e^{-\frac{t^2}{\xi}} dt \right] = \\
& \frac{2^{p+1} c (\sqrt{\xi})^{-p}}{(p+1)} \omega_1 \left(F, \sqrt{\xi} \right)_p^p (\sqrt{\xi})^{3p+1} \\
& \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \int_0^{\infty} \left(\frac{t}{\sqrt{\xi}} \right)^{3p} e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2} d\frac{t}{\sqrt{\xi}} \right] = \\
& \frac{2^{p+1} c (\sqrt{\xi})^{2p+1}}{(p+1)} \omega_1 \left(F, \sqrt{\xi} \right)_p^p \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \int_0^{\infty} x^{3p} e^{-x^2} dx \right] = \\
& \frac{2^{p+1} c (\sqrt{\xi})^{2p+1}}{(p+1)} \omega_1 \left(F, \sqrt{\xi} \right)_p^p \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \left(\int_0^1 x^{3p} e^{-x^2} dx + \int_1^{\infty} x^{3p} e^{-x^2} dx \right) \right] \leq \quad (36)
\end{aligned}$$

$$\begin{aligned}
& \frac{2^{p+1}c(\sqrt{\xi})^{2p+1}}{(p+1)}\omega_1(F, \sqrt{\xi})_p^p \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \left(\int_0^1 x^{3p} dx + \int_1^\infty x^{3p} e^{-x} dx \right) \right] \leq \\
& \frac{2^{p+1}c(\sqrt{\xi})^{2p+1}}{(p+1)}\omega_1(F, \sqrt{\xi})_p^p \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \left(\frac{1}{(3p+1)} + \int_0^\infty x^{3p} e^{-x} dx \right) \right] = \\
& \frac{2^{p+1}c(\sqrt{\xi})^{2p+1}}{(p+1)}\omega_1(F, \sqrt{\xi})_p^p \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \left(\frac{1}{(3p+1)} + \Gamma(3p+1) \right) \right] = \quad (37) \\
& \frac{\left(\sum_{j=1}^r |\alpha_j| j^2 \right)^{p-1} 2^{2p+1}}{\sqrt{\pi} |\beta^2 - \alpha^2|^p (q+1)^{p-1} (p+1)} \xi^p \omega_1(F, \sqrt{\xi})_p^p \\
& \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \left(\frac{1}{(3p+1)} + \Gamma(3p+1) \right) \right] = \\
& \frac{\left(\sum_{j=1}^r |\alpha_j| j^2 \right)^{p-1} 2^{2p+1}}{\sqrt{\pi} |\beta^2 - \alpha^2|^p (q+1)^{p-1} (p+1)} \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \left(\frac{1}{(3p+1)} + \Gamma(3p+1) \right) \right] \\
& \omega_1 \left(f^{(4)} + (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f, \sqrt{\xi} \right)_p^p \xi^p, \quad (38)
\end{aligned}$$

proving the claim. ■

It follows a consequence.

Corollary 2 (to Theorem 1) *It holds*

$$\begin{aligned}
& \|W_{r,\xi}(f) - f\|_p \leq \Phi_1(\xi) + \\
& \frac{\|f''\|_p}{|\beta^2 - \alpha^2|} \left[\sum_{j=0}^r |\alpha_j| \left| e^{-\frac{\alpha^2 j^2 \xi}{16}} - e^{-\frac{\beta^2 j^2 \xi}{16}} \right| \right] + \\
& \left(\frac{\|f^{(4)}\|_p + (\alpha^2 + \beta^2) \|f''\|_p}{(\alpha\beta)^2 |\beta^2 - \alpha^2|} \right) \\
& \left| \beta^2 \sum_{j=0}^r \alpha_j \left(1 - e^{-\frac{\alpha^2 j^2 \xi}{4}} \right) - \alpha^2 \sum_{j=0}^r \alpha_j \left(1 - e^{-\frac{\beta^2 j^2 \xi}{4}} \right) \right| \rightarrow 0, \quad (39)
\end{aligned}$$

as $\xi \rightarrow 0$.

Above $\Phi_1(\xi)$ is as in (24).

Proof. Directly from (24). ■

Next comes the L_1 approximation by $W_{r,\xi}$.

Theorem 3 Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$, and let all $f, f', f'', f^{(3)}, f^{(4)} \in L_1(\mathbb{R}) \cap C(\mathbb{R})$; $\xi > 0, x \in \mathbb{R}$. Then

$$\begin{aligned} \|E_1(x)\|_1 &= \|W_{r,\xi}(f, x) - f(x) - \\ &\frac{f''(x)}{(\beta^2 - \alpha^2)} \left[\sum_{j=0}^r \alpha_j \left(e^{-\frac{\alpha^2 j^2 \xi}{16}} - e^{-\frac{\beta^2 j^2 \xi}{16}} \right) \right] - \\ &\left(\frac{f^{(4)}(x) + (\alpha^2 + \beta^2) f''(x)}{(\alpha\beta)^2 (\beta^2 - \alpha^2)} \right) \\ &\left[\beta^2 \sum_{j=0}^r \alpha_j \left(1 - e^{-\frac{\alpha^2 j^2 \xi}{4}} \right) - \alpha^2 \sum_{j=0}^r \alpha_j \left(1 - e^{-\frac{\beta^2 j^2 \xi}{4}} \right) \right] \Big\|_1 \leq \\ &\left(\frac{4}{\sqrt{\pi} |\beta^2 - \alpha^2|} \right) \left[\sum_{j=1}^r |\alpha_j| j^2 \left(\frac{7}{3} + j \frac{25}{8} \right) \right] \\ &\omega_1 \left(f^{(4)} + (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f, \sqrt{\xi} \right)_1 \cdot \xi =: \Psi_1(\xi) \rightarrow 0, \end{aligned} \quad (40)$$

as $\xi \rightarrow 0$.

Proof. We have (F is as in (25))

$$\begin{aligned} \int_{-\infty}^{\infty} |E_1(x)| dx &\stackrel{(31)}{=} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{\pi\xi}} \left| \int_{-\infty}^{\infty} R(t) e^{-\frac{t^2}{\xi}} dt \right| \right) dx \leq \\ &\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)| \frac{e^{-\frac{t^2}{\xi}}}{\sqrt{\pi\xi}} dt \right) dx \stackrel{(30)}{\leq} \frac{1}{\sqrt{\pi\xi}} \frac{2}{|\beta^2 - \alpha^2|} \\ &\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left[\sum_{j=1}^r |\alpha_j| j^2 \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)| (|t| - \theta) d\theta \right) \right] e^{-\frac{t^2}{\xi}} dt \right) dx \leq \\ &\frac{2}{\sqrt{\pi\xi} |\beta^2 - \alpha^2|} \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \right. \right. \\ &\left. \left. \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)| d\theta \right) |t| e^{-\frac{t^2}{\xi}} dt \right) dx \right] = \\ &\frac{2}{\sqrt{\pi\xi} |\beta^2 - \alpha^2|} \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(\int_{-\infty}^{\infty} \right. \right. \right. \end{aligned} \quad (41)$$

$$\begin{aligned}
& |F(x + j \operatorname{sign}(t)\theta) - F(x)| dx d\theta |t| e^{-\frac{t^2}{\xi}} dt \Big] \leq \\
& \frac{2}{\sqrt{\pi\xi} |\beta^2 - \alpha^2|} \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \omega_1 \left(F, \sqrt{\xi} \frac{j\theta}{\sqrt{\xi}} \right)_1 d\theta \right) |t| e^{-\frac{t^2}{\xi}} dt \right] \leq \\
& \frac{2}{\sqrt{\pi\xi} |\beta^2 - \alpha^2|} \omega_1 \left(F, \sqrt{\xi} \right)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(1 + \frac{j\theta}{\sqrt{\xi}} \right) d\theta \right) |t| e^{-\frac{t^2}{\xi}} dt \right] = \\
& \frac{2}{\sqrt{\pi\xi} |\beta^2 - \alpha^2|} \omega_1 \left(F, \sqrt{\xi} \right)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(|t| + \frac{j}{2\sqrt{\xi}} t^2 \right) |t| e^{-\frac{t^2}{\xi}} dt \right] = \\
& \frac{4}{\sqrt{\pi\xi} |\beta^2 - \alpha^2|} \omega_1 \left(F, \sqrt{\xi} \right)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \int_0^{\infty} \left(t + \frac{j}{2\sqrt{\xi}} t^2 \right) t e^{-\frac{t^2}{\xi}} dt \right] = \quad (42) \\
& \frac{4}{\sqrt{\pi\xi} |\beta^2 - \alpha^2|} \omega_1 \left(F, \sqrt{\xi} \right)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \left(\int_0^{\infty} t^2 e^{-\frac{t^2}{\xi}} dt + \frac{j}{2\sqrt{\xi}} \int_0^{\infty} t^3 e^{-\frac{t^2}{\xi}} dt \right) \right] = \\
& \frac{4}{\sqrt{\pi\xi} |\beta^2 - \alpha^2|} \omega_1 \left(F, \sqrt{\xi} \right)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \left(\left(\sqrt{\xi} \right)^3 \int_0^{\infty} \left(\frac{t}{\sqrt{\xi}} \right)^2 e^{-\left(\frac{t}{\sqrt{\xi}} \right)^2} d \frac{t}{\sqrt{\xi}} + \right. \right. \\
& \quad \left. \left. \frac{j}{2\sqrt{\xi}} \left(\sqrt{\xi} \right)^4 \int_0^{\infty} \left(\frac{t}{\sqrt{\xi}} \right)^3 e^{-\left(\frac{t}{\sqrt{\xi}} \right)^2} d \frac{t}{\sqrt{\xi}} \right) \right] = \\
& \frac{4\xi}{\sqrt{\pi} |\beta^2 - \alpha^2|} \omega_1 \left(F, \sqrt{\xi} \right)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \left(\int_0^{\infty} x^2 e^{-x^2} dx + \right. \quad (43) \\
& \quad \left. \frac{j}{2} \int_0^{\infty} x^3 e^{-x^2} dx \right) \right] \leq \\
& \frac{4\xi}{\sqrt{\pi} |\beta^2 - \alpha^2|} \omega_1 \left(F, \sqrt{\xi} \right)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \left[\left(\int_0^1 x^2 dx + \int_0^{\infty} x^2 e^{-x} dx \right) \right. \right. \\
& \quad \left. \left. + \frac{j}{2} \left(\int_0^1 x^3 dx + \int_0^{\infty} x^3 e^{-x} dx \right) \right] \right] = \\
& \frac{4\xi}{\sqrt{\pi} |\beta^2 - \alpha^2|} \omega_1 \left(F, \sqrt{\xi} \right)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \left[\left(\frac{1}{3} + \Gamma(3) \right) + \frac{j}{2} \left(\frac{1}{4} + \Gamma(4) \right) \right] \right] = \\
& \frac{4\xi}{\sqrt{\pi} |\beta^2 - \alpha^2|} \omega_1 \left(F, \sqrt{\xi} \right)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \left(\frac{7}{3} + j \frac{25}{8} \right) \right] = \quad (44)
\end{aligned}$$

$$\frac{4\xi}{\sqrt{\pi}|\beta^2 - \alpha^2|} \omega_1 \left(f^{(4)} + (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f, \sqrt{\xi} \right)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \left(\frac{7}{3} + j \frac{25}{8} \right) \right],$$

The claim is proved. ■

We give

Corollary 4 (to Theorem 3) It holds

$$\begin{aligned} \|W_{r,\xi}(f) - f\|_1 &\leq \Psi_1(\xi) + \\ &\frac{\|f''\|_1}{|\beta^2 - \alpha^2|} \left[\sum_{j=0}^r |\alpha_j| \left(e^{-\frac{\alpha^2 j^2 \xi}{16}} - e^{-\frac{\beta^2 j^2 \xi}{16}} \right) \right] + \\ &\left(\frac{\|f^{(4)}\|_1 + (\alpha^2 + \beta^2) \|f''\|_1}{(\alpha\beta)^2 |\beta^2 - \alpha^2|} \right) \\ &\left| \beta^2 \sum_{j=0}^r \alpha_j \left(1 - e^{-\frac{\alpha^2 j^2 \xi}{4}} \right) - \alpha^2 \sum_{j=0}^r \alpha_j \left(1 - e^{-\frac{\beta^2 j^2 \xi}{4}} \right) \right| \rightarrow 0, \end{aligned} \quad (45)$$

as $\xi \rightarrow 0$.

Above $\Psi_1(\xi)$ is as in (40).

Proof. Directly from (40). ■

3 Part II: On the smooth Poisson-Cauchy singular integral operators ([5])

Let $\bar{\alpha} \in \mathbb{N}$, $\bar{\beta} > \frac{1}{2\bar{\alpha}}$ and $f \in C^2(\mathbb{R})$. We define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue integral

$$M_{r,\xi}(f; x) = W \int_{-\infty}^{\infty} \frac{\sum_{j=0}^r \alpha_j f(x + jt)}{\left(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}} \right)^{\bar{\beta}}} dt, \quad (46)$$

where the constant is defined as

$$W = \frac{\Gamma(\bar{\beta}) \bar{\alpha} \xi^{2\bar{\alpha}\bar{\beta}-1}}{\Gamma\left(\frac{1}{2\bar{\alpha}}\right) \Gamma\left(\bar{\beta} - \frac{1}{2\bar{\alpha}}\right)}. \quad (47)$$

We assume that $M_{r,\xi}(f; x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. We will use also that

$$M_{r,\xi}(f; x) = W \sum_{j=0}^r \alpha_j \left(\int_{-\infty}^{\infty} f(x + jt) \frac{1}{\left(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}} \right)^{\bar{\beta}}} dt \right). \quad (48)$$

We notice by $W \int_{-\infty}^{\infty} \frac{1}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} dt = 1$ that $M_{r,\xi}(c; x) = c$, c constant, and

$$M_{r,\xi}(f; x) - f(x) = W \left(\sum_{j=0}^r \alpha_j \int_{-\infty}^{\infty} [f(x+jt) - f(x)] \frac{1}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} dt \right). \quad (49)$$

We set

$$\begin{aligned} E_2(x) &:= M_{r,\xi}(f, x) - f(x) - \left(\frac{f'(x)}{\alpha\beta(\beta^2 - \alpha^2)} \right) \\ &\left[\beta^3 \sum_{j=0}^r \alpha_j W \left(\int_{-\infty}^{\infty} \sin(\alpha jt) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) - \right. \\ &\left. \alpha^3 \sum_{j=0}^r \alpha_j W \left(\int_{-\infty}^{\infty} \sin(\beta jt) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) \right] - \left(\frac{f''(x)}{\beta^2 - \alpha^2} \right) \quad (50) \\ &\left[\sum_{j=0}^r \alpha_j W \left(\int_{-\infty}^{\infty} \cos(\alpha jt) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) - \right. \\ &\left. \sum_{j=0}^r \alpha_j W \left(\int_{-\infty}^{\infty} \cos(\beta jt) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) \right] - \left(\frac{f^{(3)}(x)}{\alpha\beta(\beta^2 - \alpha^2)} \right) \\ &\left[\beta \sum_{j=0}^r \alpha_j W \left(\int_{-\infty}^{\infty} \sin(\alpha jt) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) - \right. \\ &\left. \alpha \sum_{j=0}^r \alpha_j W \left(\int_{-\infty}^{\infty} \sin(\beta jt) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) \right] - \\ &\quad \left(\frac{2(f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x))}{(\alpha\beta)^2(\beta^2 - \alpha^2)} \right) \\ &\left[\beta^2 \sum_{j=0}^r \alpha_j W \left(\int_{-\infty}^{\infty} \sin^2\left(\frac{\alpha jt}{2}\right) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) - \right. \end{aligned}$$

$$\begin{aligned}
& \alpha^2 \sum_{j=0}^r \alpha_j W \left(\int_{-\infty}^{\infty} \sin^2 \left(\frac{\beta jt}{2} \right) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) \Bigg] \\
& = M_{r,\xi}(f, x) - f(x) - \left(\frac{2f''(x)}{\beta^2 - \alpha^2} \right) W \\
& \left[\sum_{j=0}^r \alpha_j \left(\int_0^{\infty} (\cos(\alpha jt) - \cos(\beta jt)) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) \right] \\
& - \left(\frac{4(f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x))}{(\alpha\beta)^2(\beta^2 - \alpha^2)} \right) W \tag{51}
\end{aligned}$$

$$\begin{aligned}
& \left[\beta^2 \sum_{j=0}^r \alpha_j \left(\int_0^{\infty} \sin^2 \left(\frac{\alpha jt}{2} \right) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) - \right. \\
& \left. \alpha^2 \sum_{j=0}^r \alpha_j \left(\int_0^{\infty} \sin^2 \left(\frac{\beta jt}{2} \right) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) \right] \\
& = W \int_{-\infty}^{\infty} R(t) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}}. \tag{52}
\end{aligned}$$

It follows a parametrized L_p ($p > 1$) approximation result for $M_{r,\xi}$, $\xi > 0$.

Theorem 5 *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$, and let all $f, f', f'', f^{(3)}, f^{(4)} \in L_p(\mathbb{R}) \cap C(\mathbb{R})$, $1 < p < \infty$. Here $q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; $\xi > 0$, $x \in \mathbb{R}$. Assume $\bar{\alpha} \in \mathbb{N}$, $\bar{\beta} > \frac{3p+1}{2\bar{\alpha}}$. Then*

$$\begin{aligned}
& \|E_2(x)\|_p = \|M_{r,\xi}(f, x) - f(x) - \\
& \left(\frac{2f''(x)}{\beta^2 - \alpha^2} \right) W \left[\sum_{j=0}^r \alpha_j \left(\int_0^{\infty} (\cos(\alpha jt) - \cos(\beta jt)) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) \right] - \\
& \left(\frac{4(f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x))}{(\alpha\beta)^2(\beta^2 - \alpha^2)} \right) W \\
& \left[\beta^2 \sum_{j=0}^r \alpha_j \left(\int_0^{\infty} \sin^2 \left(\frac{\alpha jt}{2} \right) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) - \right.
\end{aligned}$$

$$\alpha^2 \sum_{j=0}^r \alpha_j \left(\int_0^\infty \sin^2 \left(\frac{\beta j t}{2} \right) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) \Bigg\| \Bigg\|_p \leq \quad (53)$$

$$\left[\frac{4\Gamma^{\frac{1}{p}} \left(\frac{3p+1}{2\bar{\alpha}} \right) \Gamma^{\frac{1}{p}} \left(\bar{\beta} - \left(\frac{3p+1}{2\bar{\alpha}} \right) \right)}{(p+1)^{\frac{1}{p}} (q+1)^{\frac{1}{q}} |\beta^2 - \alpha^2| \Gamma^{\frac{1}{p}} \left(\frac{1}{2\bar{\alpha}} \right) \Gamma^{\frac{1}{p}} \left(\bar{\beta} - \frac{1}{2\bar{\alpha}} \right)} \right]$$

$$\left(\sum_{j=1}^r |\alpha_j| j^2 \right)^{\frac{1}{q}} \left(\sum_{j=1}^r |\alpha_j| j^{p+2} \right)^{\frac{1}{p}}$$

$$\omega_1 \left(f^{(4)} + (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f, \xi \right)_p \cdot \xi^2 =: \Phi_2(\xi) \rightarrow 0,$$

as $\xi \rightarrow 0$.

Proof. By (52), we have $(\bar{\alpha} \in \mathbb{N}, \bar{\beta} > \frac{3p+1}{2\bar{\alpha}})$

$$E_2(x) = W \int_{-\infty}^{\infty} R(t) \frac{1}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} dt.$$

Hence it holds $(p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1)$,

$$\int_{-\infty}^{\infty} |E_2(x)|^p dx = \int_{-\infty}^{\infty} \left(W \left| \int_{-\infty}^{\infty} R(t) \frac{1}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} dt \right| \right)^p dx \leq$$

$$\int_{-\infty}^{\infty} \left(W \int_{-\infty}^{\infty} |R(t)| \frac{1}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} dt \right)^p dx \leq \quad (54)$$

$$W \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)|^p \frac{1}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} dt \right) dx \stackrel{(30)}{\leq} \frac{2^p W}{|\beta^2 - \alpha^2|^p}$$

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left[\sum_{j=1}^r |\alpha_j| j^2 \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)| (|t| - \theta) d\theta \right) \right]^p \frac{1}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} dt \right) dx$$

(by Jensen inequality)

$$\begin{aligned}
&\leq \frac{2^p W}{|\beta^2 - \alpha^2|^p} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(\sum_{j=1}^r |\alpha_j| j^2 \right)^{p-1} \right. \\
&\quad \left. \left[\sum_{j=1}^r |\alpha_j| j^2 \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)| (|t| - \theta) d\theta \right)^p \right] \frac{1}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} dt \right) dx \\
&\leq W \left(\sum_{j=1}^r |\alpha_j| j^2 \right)^{p-1} \frac{2^p}{|\beta^2 - \alpha^2|^p} \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \right. \right. \\
&\quad \left. \left. \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)|^p d\theta \right) \left(\int_0^{|t|} (|t| - \theta)^q d\theta \right)^{\frac{p}{q}} \frac{1}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} dt \right) dx \right] = \\
&\quad \quad \quad (55) \\
&\quad W \left(\sum_{j=1}^r |\alpha_j| j^2 \right)^{p-1} \frac{2^p}{|\beta^2 - \alpha^2|^p (q+1)^{p-1}} \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \right. \right. \\
&\quad \left. \left. \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)|^p d\theta \right) |t|^{(q+1)(p-1)} \frac{1}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} dt \right) dx \right] = \\
&\quad \text{(call } \rho := W \left(\sum_{j=1}^r |\alpha_j| j^2 \right)^{p-1} \frac{2^p}{|\beta^2 - \alpha^2|^p (q+1)^{p-1}}) \\
&\quad \quad \rho \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(\int_{-\infty}^{\infty} \right. \right. \right. \\
&\quad \quad \left. \left. \left. |F(x + j \operatorname{sign}(t)\theta) - F(x)|^p dx \right) d\theta \right) |t|^{(q+1)(p-1)} \frac{1}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} dt \right] \leq \\
&\quad \quad \quad \left. \left. \left. \rho \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \omega_1 \left(F, \xi \frac{j\theta}{\xi} \right)_p^p d\theta \right) |t|^{(q+1)(p-1)} \frac{1}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} dt \right] \leq \right. \right. \\
&\quad \quad \quad \left. \left. \right. \right] (56)
\end{aligned}$$

$$\begin{aligned}
& \rho \omega_1(F, \xi)_p^p \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(1 + \frac{j\theta}{\xi}\right)^p d\theta \right) |t|^{(q+1)(p-1)} \frac{1}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} dt \right] = \\
& \rho \omega_1(F, \xi)_p^p \frac{\xi}{(p+1)} \left[\sum_{j=1}^r |\alpha_j| j \int_{-\infty}^{\infty} \left[\left(1 + \frac{j}{\xi} |t|\right)^{p+1} - 1 \right] |t|^{(q+1)(p-1)} \frac{1}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} dt \right] \leq \\
& \frac{2^p \rho \xi \omega_1(F, \xi)_p^p}{(p+1)} \left[\sum_{j=1}^r |\alpha_j| j \frac{j^{p+1}}{\xi^{p+1}} \int_{-\infty}^{\infty} |t|^{p+1} |t|^{(q+1)(p-1)} \frac{1}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} dt \right] = \\
& \frac{2^{p+1} \rho \xi^{-p} \omega_1(F, \xi)_p^p}{(p+1)} \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \int_0^{\infty} t^{(q+1)(p-1)+(p+1)} \frac{1}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} dt \right] = \\
& \frac{2^{p+1} \rho \xi^{-p} \omega_1(F, \xi)_p^p}{(p+1)} \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \xi^{3p-2\bar{\alpha}\bar{\beta}+1} \int_0^{\infty} \left(\frac{t}{\xi}\right)^{3p} \frac{1}{\left(\left(\frac{t}{\xi}\right)^{2\bar{\alpha}} + 1\right)^{\bar{\beta}}} d\left(\frac{t}{\xi}\right) \right] = \\
& \frac{2^{p+1} \rho \xi^{2p-2\bar{\alpha}\bar{\beta}+1} \omega_1(F, \xi)_p^p}{(p+1)} \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \int_0^{\infty} x^{3p} \frac{1}{(x^{2\bar{\alpha}} + 1)^{\bar{\beta}}} dx \right] =
\end{aligned} \tag{57}$$

(by [13], p. 397 formula 595)

$$\begin{aligned}
& \frac{2^{p+1} \rho \xi^{2p-2\bar{\alpha}\bar{\beta}+1} \omega_1(F, \xi)_p^p}{(p+1)} \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \right] \left[\frac{\Gamma\left(\frac{3p+2}{2\bar{\alpha}}\right) \Gamma\left(\bar{\beta} - \left(\frac{3p+1}{2\bar{\alpha}}\right)\right)}{2\bar{\alpha}\Gamma(\bar{\beta})} \right] = \\
& \frac{2^{p+1} \xi^{2p-2\bar{\alpha}\bar{\beta}+1}}{(p+1)} \left(W \left(\sum_{j=1}^r |\alpha_j| j^2 \right)^{p-1} \frac{2^p}{|\beta^2 - \alpha^2|^p (q+1)^{p-1}} \right) \omega_1(F, \xi)_p^p \\
& \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \right] \left[\frac{\Gamma\left(\frac{3p+2}{2\bar{\alpha}}\right) \Gamma\left(\bar{\beta} - \left(\frac{3p+1}{2\bar{\alpha}}\right)\right)}{2\bar{\alpha}\Gamma(\bar{\beta})} \right] \tag{47} \\
& \frac{2^{p+1} \xi^{2p-2\bar{\alpha}\bar{\beta}+1}}{(q+1)^{p-1} (p+1) |\beta^2 - \alpha^2|^p} \left(\frac{\Gamma(\bar{\beta}) \bar{\alpha} \xi^{2\bar{\alpha}\bar{\beta}-1}}{\Gamma\left(\frac{1}{2\bar{\alpha}}\right) \Gamma\left(\bar{\beta} - \frac{1}{2\bar{\alpha}}\right)} \right) \left(\sum_{j=1}^r |\alpha_j| j^2 \right)^{p-1} \tag{59}
\end{aligned}$$

$$\begin{aligned}
& \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \right] \left(\frac{\Gamma\left(\frac{3p+2}{2\alpha}\right) \Gamma\left(\bar{\beta} - \left(\frac{3p+1}{2\alpha}\right)\right)}{2\alpha \Gamma(\bar{\beta})} \right) \omega_1(F, \xi)_p^p = \\
& \frac{4^p \xi^{2p}}{(q+1)^{p-1} (p+1) |\beta^2 - \alpha^2|^p \Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\bar{\beta} - \frac{1}{2\alpha}\right)} \left(\sum_{j=1}^r |\alpha_j| j^2 \right)^{p-1} \\
& \left(\sum_{j=1}^r |\alpha_j| j^{p+2} \right) \Gamma\left(\frac{3p+1}{2\alpha}\right) \Gamma\left(\bar{\beta} - \left(\frac{3p+1}{2\alpha}\right)\right) \omega_1(F, \xi)_p^p = \\
& \left[\frac{4^p \Gamma\left(\frac{3p+1}{2\alpha}\right) \Gamma\left(\bar{\beta} - \left(\frac{3p+1}{2\alpha}\right)\right)}{(p+1)(q+1)^{(p-1)} |\beta^2 - \alpha^2|^p \Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\bar{\beta} - \frac{1}{2\alpha}\right)} \right] \left(\sum_{j=1}^r |\alpha_j| j^2 \right)^{p-1} \\
& \left(\sum_{j=1}^r |\alpha_j| j^{p+2} \right) \omega_1\left(f^{(4)} + (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f, \xi\right)_p^p \cdot \xi^{2p}, \quad (60)
\end{aligned}$$

the claim is proved. ■

Next we give a consequence of Theorem 5.

Corollary 6 (to Theorem 5) *It holds*

$$\begin{aligned}
& \|M_{r,\xi}(f) - f\|_p \leq \Phi_2(\xi) + \\
& \|f''\|_p \left(\frac{2W}{|\beta^2 - \alpha^2|} \right) \left[\sum_{j=0}^r |\alpha_j| \int_0^\infty |\cos(\alpha jt) - \cos(\beta jt)| \frac{dt}{(t^{2\alpha} + \xi^{2\alpha})^{\bar{\beta}}} \right] + \\
& \left(\frac{4 \|f^{(4)}\|_p + (\alpha^2 + \beta^2) \|f''\|_p}{(\alpha\beta)^2 |\beta^2 - \alpha^2|} \right) W \quad (61) \\
& \left[\beta^2 \sum_{j=0}^r |\alpha_j| \left(\int_0^\infty \sin^2\left(\frac{\alpha jt}{2}\right) \frac{dt}{(t^{2\alpha} + \xi^{2\alpha})^{\bar{\beta}}} \right) + \right. \\
& \left. \alpha^2 \sum_{j=0}^r |\alpha_j| \left(\int_0^\infty \sin^2\left(\frac{\beta jt}{2}\right) \frac{dt}{(t^{2\alpha} + \xi^{2\alpha})^{\bar{\beta}}} \right) \right] \rightarrow 0,
\end{aligned}$$

as $\xi \rightarrow 0$.

Above $\Phi_2(\xi)$ is as in (53).

Proof. We have

$$\begin{aligned}
& W \int_0^\infty |\cos(\alpha jt) - \cos(\beta jt)| \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} = \\
& 2W \int_0^\infty \left| \sin\left(\frac{(\alpha + \beta) jt}{2}\right) \right| \left| \sin\left(\frac{(\alpha - \beta) jt}{2}\right) \right| \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \leq \\
& 2W \int_0^\infty \frac{|\alpha + \beta| jt}{2} \frac{|\alpha - \beta| jt}{2} \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} = \\
& \frac{W |\alpha^2 - \beta^2| j^2}{2} \int_0^\infty \frac{t^2 dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} = \frac{W |\alpha^2 - \beta^2| j^2}{2} \xi^{3-2\bar{\alpha}\bar{\beta}} \int_0^\infty \frac{x^2 dx}{(1 + x^{2\bar{\alpha}})^{\bar{\beta}}} = \\
\end{aligned} \tag{62}$$

(by [13], p. 397, formula 595)

$$\begin{aligned}
& \frac{|\alpha^2 - \beta^2| j^2}{2} \left(\frac{\Gamma(\bar{\beta}) \bar{\alpha} \xi^{2\bar{\alpha}\bar{\beta}-1}}{\Gamma(\frac{1}{2\bar{\alpha}}) \Gamma(\bar{\beta} - \frac{1}{2\bar{\alpha}})} \right) \xi^{3-2\bar{\alpha}\bar{\beta}} \frac{\Gamma(\frac{3}{2\bar{\alpha}}) \Gamma(\bar{\beta} - \frac{3}{2\bar{\alpha}})}{2\bar{\alpha}\Gamma(\bar{\beta})} = \\
& \frac{|\beta^2 - \alpha^2| j^2}{4} \frac{\Gamma(\frac{3}{2\bar{\alpha}}) \Gamma(\bar{\beta} - \frac{3}{2\bar{\alpha}})}{\Gamma(\frac{1}{2\bar{\alpha}}) \Gamma(\bar{\beta} - \frac{1}{2\bar{\alpha}})} \xi^2 \rightarrow 0, \text{ as } \xi \rightarrow 0.
\end{aligned} \tag{63}$$

Furthermore, we have that

$$\begin{aligned}
& W \int_0^\infty \sin^2\left(\frac{\alpha jt}{2}\right) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \leq W \frac{\alpha^2 j^2}{4} \int_0^\infty t^2 \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} = \\
& \left(\frac{\Gamma(\bar{\beta}) \bar{\alpha} \xi^{2\bar{\alpha}\bar{\beta}-1}}{\Gamma(\frac{1}{2\bar{\alpha}}) \Gamma(\bar{\beta} - \frac{1}{2\bar{\alpha}})} \right) \frac{\alpha^2 j^2}{4} \xi^{3-2\bar{\alpha}\bar{\beta}} \left(\int_0^\infty \frac{x^2 dx}{(1 + x^{2\bar{\alpha}})^{\bar{\beta}}} \right) = \\
& \left(\frac{\Gamma(\bar{\beta}) \bar{\alpha} \xi^2}{\Gamma(\frac{1}{2\bar{\alpha}}) \Gamma(\bar{\beta} - \frac{1}{2\bar{\alpha}})} \right) \frac{\alpha^2 j^2}{4} \frac{\Gamma(\frac{3}{2\bar{\alpha}}) \Gamma(\bar{\beta} - \frac{3}{2\bar{\alpha}})}{2\bar{\alpha}\Gamma(\bar{\beta})} = \\
& \frac{\alpha^2 j^2}{8} \left(\frac{\Gamma(\frac{3}{2\bar{\alpha}}) \Gamma(\bar{\beta} - \frac{3}{2\bar{\alpha}})}{\Gamma(\frac{1}{2\bar{\alpha}}) \Gamma(\bar{\beta} - \frac{1}{2\bar{\alpha}})} \right) \xi^2 \rightarrow 0, \text{ as } \xi \rightarrow 0.
\end{aligned} \tag{64}$$

The proof now is clear. ■

Next comes the L_1 approximation by $M_{r,\xi}$.

Theorem 7 Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$, and let all $f, f', f'', f^{(3)}, f^{(4)} \in L_1(\mathbb{R}) \cap C(\mathbb{R})$; $\xi > 0, x \in \mathbb{R}$. Here $\bar{\alpha} \in \mathbb{N}, \bar{\beta} > \frac{2}{\bar{\alpha}}$. Then

$$\|E_2(x)\|_1 = \|M_{r,\xi}(f, x) - f(x) -$$

$$\begin{aligned}
& \left(\frac{2f''(x)}{\beta^2 - \alpha^2} \right) W \left[\sum_{j=0}^r \alpha_j \left(\int_0^\infty (\cos(\alpha jt) - \cos(\beta jt)) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) \right] - \\
& \left(\frac{4(f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x))}{(\alpha\beta)^2(\beta^2 - \alpha^2)} \right) W \\
& \left[\beta^2 \sum_{j=0}^r \alpha_j \left(\int_0^\infty \sin^2\left(\frac{\alpha jt}{2}\right) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) - \right. \\
& \left. \alpha^2 \sum_{j=0}^r \alpha_j \left(\int_0^\infty \sin^2\left(\frac{\beta jt}{2}\right) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) \right] \Bigg\|_1 \leq \quad (66) \\
& \left(\frac{2}{|\beta^2 - \alpha^2| \Gamma\left(\frac{1}{2\bar{\alpha}}\right) \Gamma\left(\bar{\beta} - \frac{1}{2\bar{\alpha}}\right)} \right) \\
& \left[\sum_{j=1}^r |\alpha_j| j^2 \left[\Gamma\left(\frac{3}{2\bar{\alpha}}\right) \Gamma\left(\bar{\beta} - \frac{3}{2\bar{\alpha}}\right) + \frac{j}{2} \Gamma\left(\frac{2}{\bar{\alpha}}\right) \Gamma\left(\bar{\beta} - \frac{2}{\bar{\alpha}}\right) \right] \right] \\
& \omega_1 \left(f^{(4)} + (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f, \xi \right)_1 \cdot \xi^2 =: \Psi_2(\xi) \rightarrow 0,
\end{aligned}$$

as $\xi \rightarrow 0$.

Proof. We have that (F is as in (25))

$$\begin{aligned}
& \int_{-\infty}^\infty |E_2(x)| dx \stackrel{(52)}{=} W \int_{-\infty}^\infty \left| \int_{-\infty}^\infty R(t) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right| dx \leq \\
& W \int_{-\infty}^\infty \left(\int_{-\infty}^\infty |R(t)| \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) dx \stackrel{(30)}{\leq} \frac{2W}{|\beta^2 - \alpha^2|} \\
& \int_{-\infty}^\infty \left(\int_{-\infty}^\infty \left[\sum_{j=1}^r |\alpha_j| j^2 \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)| (|t| - \theta) d\theta \right) \right] \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) dx \leq \\
& \quad (67) \\
& \frac{2W}{|\beta^2 - \alpha^2|} \int_{-\infty}^\infty \left(\int_{-\infty}^\infty \left[\sum_{j=1}^r |\alpha_j| j^2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)| d\theta \right) |t| \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} dx \Bigg] = \\
& \frac{2W}{|\beta^2 - \alpha^2|} \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(\int_{-\infty}^{\infty} |F(x + j \operatorname{sign}(t)\theta) - F(x)| dx \right) d\theta \right) \frac{|t| dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right] \leq \\
& \frac{2W}{|\beta^2 - \alpha^2|} \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \omega_1 \left(F, \xi \frac{j\theta}{\xi} \right)_1 d\theta \right) \frac{|t| dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right] \leq \\
& \frac{2W}{|\beta^2 - \alpha^2|} \omega_1(F, \xi)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(1 + \frac{j\theta}{\xi} \right) d\theta \right) \frac{|t| dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right] = \\
& \frac{2W}{|\beta^2 - \alpha^2|} \omega_1(F, \xi)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(|t| + \frac{j}{2\xi} t^2 \right) \frac{|t| dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right] = \quad (68) \\
& \frac{4W}{|\beta^2 - \alpha^2|} \omega_1(F, \xi)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \int_0^{\infty} \left(t + \frac{j}{2\xi} t^2 \right) \frac{t dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right] = \\
& \frac{4W}{|\beta^2 - \alpha^2|} \omega_1(F, \xi)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \left[\int_0^{\infty} \frac{t^2 dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} + \frac{j}{2\xi} \int_0^{\infty} \frac{t^3 dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right] \right] = \\
& \frac{4W}{|\beta^2 - \alpha^2|} \omega_1(F, \xi)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \left[\xi^{3-2\bar{\alpha}\bar{\beta}} \int_0^{\infty} \frac{x^2 dx}{(1+x^{2\bar{\alpha}})^{\bar{\beta}}} + \right. \right. \\
& \quad \left. \left. \frac{j}{2\xi} \xi^{4-2\bar{\alpha}\bar{\beta}} \int_0^{\infty} \frac{x^3 dx}{(1+x^{2\bar{\alpha}})^{\bar{\beta}}} \right] \right] = \quad (69) \\
& \frac{4W \xi^{3-2\bar{\alpha}\bar{\beta}}}{|\beta^2 - \alpha^2|} \omega_1(F, \xi)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \left[\int_0^{\infty} \frac{x^2 dx}{(1+x^{2\bar{\alpha}})^{\bar{\beta}}} + \right. \right.
\end{aligned}$$

$$\left. \left. \frac{j}{2} \int_0^\infty \frac{x^3 dx}{(1+x^{2\bar{\alpha}})^{\bar{\beta}}} \right] \right] =$$

(by [13], p. 397, formula 595)

$$\begin{aligned} & \frac{4}{|\beta^2 - \alpha^2|} \left(\frac{\Gamma(\bar{\beta}) \bar{\alpha} \xi^2}{\Gamma(\frac{1}{2\bar{\alpha}}) \Gamma(\bar{\beta} - \frac{1}{2\bar{\alpha}})} \right) \omega_1(F, \xi)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \left[\frac{\Gamma(\frac{3}{2\bar{\alpha}}) \Gamma(\bar{\beta} - \frac{3}{2\bar{\alpha}})}{2\bar{\alpha} \Gamma(\bar{\beta})} \right. \right. \\ & \left. \left. + \frac{j}{2} \frac{\Gamma(\frac{2}{\bar{\alpha}}) \Gamma(\bar{\beta} - \frac{2}{\bar{\alpha}})}{2\bar{\alpha} \Gamma(\bar{\beta})} \right] \right] = \tag{70} \\ & \frac{2}{|\beta^2 - \alpha^2| \Gamma(\frac{1}{2\bar{\alpha}}) \Gamma(\bar{\beta} - \frac{1}{2\bar{\alpha}})} \omega_1(F, \xi)_1 \xi^2 \\ & \left[\sum_{j=1}^r |\alpha_j| j^2 \left[\Gamma\left(\frac{3}{2\bar{\alpha}}\right) \Gamma\left(\bar{\beta} - \frac{3}{2\bar{\alpha}}\right) + \frac{j}{2} \Gamma\left(\frac{2}{\bar{\alpha}}\right) \Gamma\left(\bar{\beta} - \frac{2}{\bar{\alpha}}\right) \right] \right]. \end{aligned}$$

The claim is proved. ■

It follows the related result.

Corollary 8 (to Theorem 7) *It holds*

$$\begin{aligned} & \|M_{r,\xi}(f) - f\|_1 \leq \Psi_2(\xi) + \|f''\|_1 \left(\frac{2W}{|\beta^2 - \alpha^2|} \right) \\ & \left[\sum_{j=0}^r |\alpha_j| \int_0^\infty |\cos(\alpha jt) - \cos(\beta jt)| \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right] + \\ & \left(\frac{4 \|f^{(4)}\|_1 + (\alpha^2 + \beta^2) \|f''\|_1}{(\alpha\beta)^2 |\beta^2 - \alpha^2|} \right) W \tag{71} \\ & \left[\beta^2 \sum_{j=0}^r |\alpha_j| \left(\int_0^\infty \sin^2\left(\frac{\alpha jt}{2}\right) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) + \right. \\ & \left. \alpha^2 \sum_{j=0}^r |\alpha_j| \left(\int_0^\infty \sin^2\left(\frac{\beta jt}{2}\right) \frac{dt}{(t^{2\bar{\alpha}} + \xi^{2\bar{\alpha}})^{\bar{\beta}}} \right) \right] \rightarrow 0 \end{aligned}$$

as $\xi \rightarrow 0$. Above $\Psi_2(\xi)$ is as in (66).

Proof. Directly from (66), and see also the proof of Corollary 6. ■

4 Part III: On the smooth Trigonometric singular integral operators ([5])

Let $\xi > 0$, $f \in C^2(\mathbb{R})$, $x \in \mathbb{R}$, $\bar{\beta} \in \mathbb{N}$; we set

$$T_{r,\xi}(f; x) := \frac{1}{\lambda} \int_{-\infty}^{\infty} \left(\sum_{j=0}^r \alpha_j f(x + jt) \right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt, \quad (72)$$

where

$$\lambda := \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt = 2\xi^{1-2\bar{\beta}} \int_0^{\infty} \left(\frac{\sin t}{t} \right)^{2\bar{\beta}} dt \quad (73)$$

(by [11], p. 210, item 1033)

$$= 2\xi^{1-2\bar{\beta}} \pi (-1)^{\bar{\beta}} \bar{\beta} \sum_{k=1}^{\bar{\beta}} (-1)^k \frac{k^{2\bar{\beta}-1}}{(\bar{\beta}-k)! (\bar{\beta}+k)!}.$$

Denote

$$\lambda_1 := 2\pi (-1)^{\bar{\beta}} \bar{\beta} \sum_{k=1}^{\bar{\beta}} (-1)^k \frac{k^{2\bar{\beta}-1}}{(\bar{\beta}-k)! (\bar{\beta}+k)!}; \quad (74)$$

that is

$$\lambda = \lambda_1 \xi^{1-2\bar{\beta}}. \quad (75)$$

We suppose that $T_{r,\xi}(f; x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. Clearly, again it is

$$\frac{1}{\lambda} \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt = 1, \quad (76)$$

and $T_{r,\xi}(c; x) = c$, c constant, and

$$T_{r,\xi}(f; x) - f(x) = \frac{1}{\lambda} \left(\sum_{j=0}^r \alpha_j \int_{-\infty}^{\infty} [f(x + jt) - f(x)] \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right). \quad (77)$$

We set

$$E_3(x) := T_{r,\xi}(f, x) - f(x) - \left(\frac{f'(x)}{\alpha\beta(\beta^2 - \alpha^2)} \right) \\ \left[\beta^3 \sum_{j=0}^r \alpha_j \frac{1}{\lambda} \left(\int_{-\infty}^{\infty} \sin(\alpha jt) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) - \right.$$

$$\begin{aligned}
& \alpha^3 \sum_{j=0}^r \alpha_j \frac{1}{\lambda} \left(\int_{-\infty}^{\infty} \sin(\beta jt) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) - \\
& \left(\frac{f''(x)}{\beta^2 - \alpha^2} \right) \left[\sum_{j=0}^r \alpha_j \frac{1}{\lambda} \left(\int_{-\infty}^{\infty} \cos(\alpha jt) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) - \right. \\
& \left. \sum_{j=0}^r \alpha_j \frac{1}{\lambda} \left(\int_{-\infty}^{\infty} \cos(\beta jt) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) \right] - \\
& \left(\frac{f^{(3)}(x)}{\alpha\beta(\beta^2 - \alpha^2)} \right) \left[\beta \sum_{j=0}^r \alpha_j \frac{1}{\lambda} \left(\int_{-\infty}^{\infty} \sin(\alpha jt) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) - \right. \quad (78)
\end{aligned}$$

$$\begin{aligned}
& \left. \alpha \sum_{j=0}^r \alpha_j \frac{1}{\lambda} \left(\int_{-\infty}^{\infty} \sin(\beta jt) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) \right] - \\
& \left(\frac{2(f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x))}{(\alpha\beta)^2(\beta^2 - \alpha^2)} \right) \\
& \left[\beta^2 \sum_{j=0}^r \alpha_j \frac{1}{\lambda} \left(\int_{-\infty}^{\infty} \sin^2\left(\frac{\alpha jt}{2}\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) - \right. \\
& \left. \alpha^2 \sum_{j=0}^r \alpha_j \frac{1}{\lambda} \left(\int_{-\infty}^{\infty} \sin^2\left(\frac{\beta jt}{2}\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) \right] \\
& = T_{r,\xi}(f, x) - f(x) - \left(\frac{2f''(x)}{\beta^2 - \alpha^2} \right) \frac{1}{\lambda} \\
& \left[\sum_{j=0}^r \alpha_j \left(\int_{-\infty}^{\infty} (\cos(\alpha jt) - \cos(\beta jt)) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) \right] - \\
& \left(\frac{4(f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x))}{(\alpha\beta)^2(\beta^2 - \alpha^2)} \right) \frac{1}{\lambda} \quad (79)
\end{aligned}$$

$$\begin{aligned}
& \left[\beta^2 \sum_{j=0}^r \alpha_j \left(\int_{-\infty}^{\infty} \sin^2 \left(\frac{\alpha jt}{2} \right) \left(\frac{\sin \left(\frac{t}{\xi} \right)}{t} \right)^{2\bar{\beta}} dt \right) - \right. \\
& \left. \alpha^2 \sum_{j=0}^r \alpha_j \left(\int_{-\infty}^{\infty} \sin^2 \left(\frac{\beta jt}{2} \right) \left(\frac{\sin \left(\frac{t}{\xi} \right)}{t} \right)^{2\bar{\beta}} dt \right) \right] \\
& = \frac{1}{\lambda} \int_{-\infty}^{\infty} R(t) \left(\frac{\sin \left(\frac{t}{\xi} \right)}{t} \right)^{2\bar{\beta}} dt. \tag{80}
\end{aligned}$$

It follows a parametrized L_p ($p > 1$) approximation result for $T_{r,\xi}$, $\xi > 0$.

Theorem 9 Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\bar{\beta} \in \mathbb{N}$ and $\bar{\beta} > \frac{[3p]+1}{2}$ ($[\cdot]$ the ceiling of the number), $\xi > 0$, λ_1 is as in (74).

When $\bar{\lambda} \in \mathbb{N}$ is even we define

$$\psi_{1\bar{\lambda}} := \frac{\pi (-1)^{\frac{2\bar{\beta}-\bar{\lambda}}{2}} (2\bar{\beta})!}{2^{\bar{\lambda}+1} (2\bar{\beta} - \bar{\lambda} - 1)!} \left(\sum_{k=1}^{\bar{\beta}} (-1)^k \frac{k^{2\bar{\beta}-\bar{\lambda}-1}}{(\bar{\beta}-k)! (\bar{\beta}+k)!} \right),$$

and when $\bar{\lambda}$ is odd we define

$$\psi_{2\bar{\lambda}} := \frac{(-1)^{\frac{\bar{\lambda}-1}{2}} (2\bar{\beta})!}{2^{\bar{\lambda}} (2\bar{\beta} - \bar{\lambda} - 1)!} \left(\sum_{k=1}^{\bar{\beta}} (-1)^{\bar{\beta}-k} \frac{k^{2\bar{\beta}-\bar{\lambda}-1} \ln(2k)}{(\bar{\beta}-k)! (\bar{\beta}+k)!} \right),$$

and we set

$$\psi_{\bar{\lambda}} := \begin{cases} \psi_{1\bar{\lambda}}, & \text{if } \bar{\lambda} \text{ is even,} \\ \psi_{2\bar{\lambda}}, & \text{if } \bar{\lambda} \text{ is odd} \end{cases}.$$

Let $\alpha, \beta \in \mathbb{R} : \alpha\beta(\alpha^2 - \beta^2) \neq 0$; $f, f', f'', f^{(3)}, f^{(4)} \in L_p(\mathbb{R}) \cap C(\mathbb{R})$; $\xi > 0$, $x \in \mathbb{R}$.

Then

$$\begin{aligned}
\|E_3\|_p &= \left\| T_{r,\xi}(f, x) - f(x) - \left(\frac{2f''(x)}{\beta^2 - \alpha^2} \right) \frac{1}{\lambda} \right. \\
& \left[\sum_{j=0}^r \alpha_j \left(\int_0^{\infty} (\cos(\alpha jt) - \cos(\beta jt)) \left(\frac{\sin \left(\frac{t}{\xi} \right)}{t} \right)^{2\bar{\beta}} dt \right) \right] - \\
& \left(\frac{4(f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x))}{(\alpha\beta)^2(\beta^2 - \alpha^2)} \right) \frac{1}{\lambda}
\end{aligned}$$

$$\begin{aligned}
& \left[\beta^2 \sum_{j=0}^r \alpha_j \left(\int_0^\infty \sin^2 \left(\frac{\alpha_j t}{2} \right) \left(\frac{\sin \left(\frac{t}{\xi} \right)}{t} \right)^{2\bar{\beta}} dt \right) - \right. \\
& \left. \alpha^2 \sum_{j=0}^r \alpha_j \left(\int_0^\infty \sin^2 \left(\frac{\beta_j t}{2} \right) \left(\frac{\sin \left(\frac{t}{\xi} \right)}{t} \right)^{2\bar{\beta}} dt \right) \right] \Bigg\|_p \leq \quad (81) \\
& \leq \frac{4 \cdot 2^{\frac{1}{p}}}{\lambda_1^{\frac{1}{p}} |\beta^2 - \alpha^2| (q+1)^{\frac{1}{q}} (p+1)^{\frac{1}{p}}} \left(\sum_{j=1}^r |\alpha_j| j^2 \right)^{\frac{1}{q}} \left(\sum_{j=1}^r |\alpha_j| j^{p+2} \right)^{\frac{1}{p}} \\
& \quad \left[\left(\frac{-(2\bar{\beta})!}{8(2\bar{\beta}-4)!} \sum_{k=1}^{\bar{\beta}} (-1)^{\bar{\beta}-k} \frac{k^{2\bar{\beta}-4} \ln(2k)}{(\bar{\beta}-k)! (\bar{\beta}+k)!} \right) + \psi_{[3p]} \right]^{\frac{1}{p}} \\
& \omega_1 \left(f^{(4)} + (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f, \xi \right)_p \cdot \xi^2 =: \Phi_3(\xi) \rightarrow 0, \quad \text{as } \xi \rightarrow 0.
\end{aligned}$$

Proof. By (80) we have $(\bar{\beta} \in \mathbb{N} : \bar{\beta} > \frac{[3p]+1}{2})$

$$E_3(x) = \frac{1}{\lambda} \int_{-\infty}^{\infty} R(t) \left(\frac{\sin \left(\frac{t}{\xi} \right)}{t} \right)^{2\bar{\beta}} dt. \quad (82)$$

Hence it holds $(p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1)$

$$\begin{aligned}
\int_{-\infty}^{\infty} |E_3(x)|^p dx &= \int_{-\infty}^{\infty} \left(\frac{1}{\lambda} \left| \int_{-\infty}^{\infty} R(t) \left(\frac{\sin \left(\frac{t}{\xi} \right)}{t} \right)^{2\bar{\beta}} dt \right| \right)^p dx \leq \\
& \int_{-\infty}^{\infty} \left(\frac{1}{\lambda} \int_{-\infty}^{\infty} |R(t)| \left(\frac{\sin \left(\frac{t}{\xi} \right)}{t} \right)^{2\bar{\beta}} dt \right)^p dx \leq \quad (83) \\
& \frac{1}{\lambda} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)|^p \left(\frac{\sin \left(\frac{t}{\xi} \right)}{t} \right)^{2\bar{\beta}} dt \right) dx \stackrel{(30)}{\leq} \\
& \frac{2^p}{|\beta^2 - \alpha^2|^p \lambda} \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left[\sum_{j=0}^r |\alpha_j| j^2 \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)| (|t| - \theta) d\theta \right)^p \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \Big) dx \\
& \stackrel{\text{(by Jensen's inequality)}}{\leq} \frac{2^p}{|\beta^2 - \alpha^2|^p \lambda} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(\sum_{j=0}^r |\alpha_j| j^2 \right)^{p-1} \right. \\
& \left. \left[\sum_{j=0}^r |\alpha_j| j^2 \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)| (|t| - \theta) d\theta \right)^p \right] \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) dx \leq \\
& \frac{1}{\lambda} \left(\sum_{j=0}^r |\alpha_j| j^2 \right)^{p-1} \frac{2^p}{|\beta^2 - \alpha^2|^p} \\
& \left[\sum_{j=0}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)|^p d\theta \right) \right. \right. \\
& \left. \left. \left(\int_0^{|t|} (|t| - \theta)^q d\theta \right)^{\frac{p}{q}} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) dx \right] = \tag{84} \\
& \frac{1}{\lambda} \left(\sum_{j=0}^r |\alpha_j| j^2 \right)^{p-1} \frac{2^p}{|\beta^2 - \alpha^2|^p (q+1)^{p-1}} \\
& \left[\sum_{j=0}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)|^p d\theta \right) \right. \right. \\
& \left. \left. |t|^{(q+1)(p-1)} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) dx \right] \\
& \text{(call } \mu := \frac{1}{\lambda} \left(\sum_{j=0}^r |\alpha_j| j^2 \right)^{p-1} \frac{2^p}{|\beta^2 - \alpha^2|^p (q+1)^{p-1}} \text{)} \\
& = \mu \left[\sum_{j=0}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(\int_{-\infty}^{\infty} |F(x + j \operatorname{sign}(t)\theta) - F(x)|^p dx \right) d\theta \right) \right]
\end{aligned}$$

$$\begin{aligned}
& \left[|t|^{(q+1)(p-1)} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right] \leq \\
\mu & \left[\sum_{j=0}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \omega_1 \left(F, \xi \frac{j\theta}{\xi} \right)_p^p d\theta \right) |t|^{(q+1)(p-1)} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right] \leq \\
& \mu \omega_1(F, \xi)_p^p \\
& \left[\sum_{j=0}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(1 + \frac{j\theta}{\xi} \right)^p d\theta \right) |t|^{(q+1)(p-1)} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right] = \\
& \mu \omega_1(F, \xi)_p^p \frac{\xi}{(p+1)} \\
& \left[\sum_{j=1}^r |\alpha_j| j \int_{-\infty}^{\infty} \left(\left(1 + \frac{j}{\xi} |t| \right)^{p+1} - 1 \right) |t|^{(q+1)(p-1)} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right] \leq \\
& \frac{2^p \mu \xi \omega_1(F, \xi)_p^p}{(p+1)} \\
& \left[\sum_{j=1}^r |\alpha_j| j \frac{j^{p+1}}{\xi^{p+1}} \int_{-\infty}^{\infty} |t|^{p+1} |t|^{(q+1)(p-1)} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right] = \quad (86) \\
& \frac{2^{p+1} \mu \xi^{-p} \omega_1(F, \xi)_p^p}{(p+1)} \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \right. \\
& \left. \int_0^{\infty} t^{(q+1)(p-1)+(p+1)} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right] = \\
& \frac{2^{p+1} \mu \xi^{-p} \omega_1(F, \xi)_p^p}{(p+1)} \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \xi^{3p-2\bar{\beta}+1} \right. \\
& \left. \int_0^{\infty} \left(\frac{t}{\xi} \right)^{3p} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{\left(\frac{t}{\xi}\right)} \right)^{2\bar{\beta}} d\frac{t}{\xi} \right] =
\end{aligned}$$

$$\frac{2^{p+1} \mu \xi^{2p-2\bar{\beta}+1} \omega_1(F, \xi)_p^p}{(p+1)} \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \right] \left[\int_0^\infty x^{3p} \left(\frac{\sin x}{x} \right)^{2\bar{\beta}} dx \right] = \quad (87)$$

$$\begin{aligned} & \frac{2^{p+1} \mu \xi^{2p-2\bar{\beta}+1} \omega_1(F, \xi)_p^p}{(p+1)} \left[\sum_{j=1}^r |\alpha_j| j^{p+2} \right] \\ & \left[\int_0^1 x^{3p} \left(\frac{\sin x}{x} \right)^{2\bar{\beta}} dx + \int_1^\infty x^{3p} \left(\frac{\sin x}{x} \right)^{2\bar{\beta}} dx \right] \leq \\ & \frac{2^{p+1} \mu \xi^{2p-2\bar{\beta}+1} \omega_1(F, \xi)_p^p}{(p+1)} \left(\sum_{j=1}^r |\alpha_j| j^{p+2} \right) \\ & \left[\int_0^1 x^3 \left(\frac{\sin x}{x} \right)^{2\bar{\beta}} dx + \int_1^\infty x^{[3p]} \left(\frac{\sin x}{x} \right)^{2\bar{\beta}} dx \right] \leq \\ & \frac{2^{p+1} \mu \xi^{2p-2\bar{\beta}+1} \omega_1(F, \xi)_p^p}{(p+1)} \left(\sum_{j=1}^r |\alpha_j| j^{p+2} \right) \quad (88) \\ & \left[\int_0^\infty x^3 \left(\frac{\sin x}{x} \right)^{2\bar{\beta}} dx + \int_0^\infty x^{[3p]} \left(\frac{\sin x}{x} \right)^{2\bar{\beta}} dx \right] \end{aligned}$$

(we use [11], p. 210, item 1033)

$$\begin{aligned} & = \frac{2^{p+1} \mu \xi^{2p-2\bar{\beta}+1} \omega_1(F, \xi)_p^p}{(p+1)} \left(\sum_{j=1}^r |\alpha_j| j^{p+2} \right) \\ & \left[\left(\frac{-(2\bar{\beta})!}{8(2\bar{\beta}-4)!} \sum_{k=1}^{\bar{\beta}} (-1)^{\bar{\beta}-k} \frac{k^{2\bar{\beta}-4} \ln(2k)}{(\bar{\beta}-k)! (\bar{\beta}+k)!} \right) + \psi_{[3p]} \right] = \\ & \frac{2^{p+1} \xi^{2p} \omega_1(F, \xi)_p^p}{(p+1)} \left(\sum_{j=1}^r |\alpha_j| j^{p+2} \right) \\ & \left[\left(\frac{-(2\bar{\beta})!}{8(2\bar{\beta}-4)!} \sum_{k=1}^{\bar{\beta}} (-1)^{\bar{\beta}-k} \frac{k^{2\bar{\beta}-4} \ln(2k)}{(\bar{\beta}-k)! (\bar{\beta}+k)!} \right) + \psi_{[3p]} \right] \\ & \frac{1}{\lambda_1} \left(\sum_{j=0}^r |\alpha_j| j^2 \right)^{p-1} \frac{2^p}{|\beta^2 - \alpha^2|^p (q+1)^{p-1}} = \quad (89) \end{aligned}$$

$$\frac{2^{2p+1}}{\lambda_1 |\beta^2 - \alpha^2|^p (q+1)^{p-1} (p+1)} \left(\sum_{j=0}^r |\alpha_j| j^2 \right)^{p-1} \left(\sum_{j=1}^r |\alpha_j| j^{p+2} \right) \\ \left[\left(\frac{-(2\bar{\beta})!}{8(2\bar{\beta}-4)!} \sum_{k=1}^{\bar{\beta}} (-1)^{\bar{\beta}-k} \frac{k^{2\bar{\beta}-4} \ln(2k)}{(\bar{\beta}-k)! (\bar{\beta}+k)!} \right) + \psi_{\lceil 3p \rceil} \right] \omega_1(F, \xi)_p^p \xi^{2p} < \infty.$$

The claim is proved. ■

Next we give a consequence of Theorem 9.

Corollary 10 (to Theorem 9) *It holds*

$$\|T_{r,\xi}(f) - f\|_p \leq \Phi_3(\xi) + \|f''\|_p \left(\frac{2\xi^{2\bar{\beta}-1}}{\lambda_1 |\beta^2 - \alpha^2|} \right) \\ \left[\sum_{j=0}^r |\alpha_j| \int_0^\infty |\cos(\alpha jt) - \cos(\beta jt)| \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right] + \\ \left(\frac{4\|f^{(4)}\|_p + (\alpha^2 + \beta^2)\|f''\|_p}{(\alpha\beta)^2 |\beta^2 - \alpha^2|} \right) \frac{\xi^{2\bar{\beta}-1}}{\lambda_1} \\ \left[\beta^2 \sum_{j=0}^r |\alpha_j| \left(\int_0^\infty \sin^2\left(\frac{\alpha jt}{2}\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) + \right. \\ \left. \alpha^2 \sum_{j=0}^r |\alpha_j| \left(\int_0^\infty \sin^2\left(\frac{\beta jt}{2}\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) \right] \rightarrow 0, \quad (90)$$

as $\xi \rightarrow 0$. Above $\Phi_3(\xi)$ is as in (81).

Proof. We have that

$$\frac{\xi^{2\bar{\beta}-1}}{\lambda_1} \int_0^\infty |\cos(\alpha jt) - \cos(\beta jt)| \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt = \\ \frac{2\xi^{2\bar{\beta}-1}}{\lambda_1} \int_0^\infty \left| \sin\left(\frac{(\alpha + \beta) jt}{2}\right) \right| \left| \sin\left(\frac{(\alpha - \beta) jt}{2}\right) \right| \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \leq \\ \frac{2\xi^{2\bar{\beta}-1}}{\lambda_1} \int_0^\infty \frac{|\alpha + \beta| jt}{2} \frac{|\alpha - \beta| jt}{2} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt = \quad (91)$$

$$\begin{aligned}
& \frac{\xi^{2\bar{\beta}-1} |\alpha^2 - \beta^2|}{\lambda_1} j^2 \int_0^\infty t^2 \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt = \\
& \frac{\xi^{2\bar{\beta}-1}}{\lambda_1} \xi^{3-2\bar{\beta}} \frac{|\alpha^2 - \beta^2|}{2} j^2 \int_0^\infty \left(\frac{t}{\xi}\right)^2 \left(\frac{\sin\left(\frac{t}{\xi}\right)}{\left(\frac{t}{\xi}\right)} \right)^{2\bar{\beta}} d\frac{t}{\xi} = \\
& \frac{\xi^2 |\alpha^2 - \beta^2|}{\lambda_1} j^2 \int_0^\infty x^2 \left(\frac{\sin x}{x} \right)^{2\bar{\beta}} dx = \frac{\xi^2 |\alpha^2 - \beta^2|}{\lambda_1} j^2 \psi_{12} \rightarrow 0, \text{ as } \xi \rightarrow 0.
\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}
\frac{\xi^{2\bar{\beta}-1}}{\lambda_1} \int_0^\infty \sin^2\left(\frac{\alpha jt}{2}\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt &\leq \frac{\xi^{2\bar{\beta}-1} \alpha^2 j^2}{\lambda_1} \int_0^\infty t^2 \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt = \\
& \frac{\xi^2 \alpha^2 j^2}{\lambda_1} \psi_{12} \rightarrow 0, \text{ as } \xi \rightarrow 0.
\end{aligned} \tag{92}$$

The proof now is clear. ■

Next comes the corresponding L_1 approximation result.

Theorem 11 *Let $\bar{\beta} \in \mathbb{N} : \bar{\beta} > 2$, and $\alpha, \beta \in \mathbb{R} : \alpha\beta(\alpha^2 - \beta^2) \neq 0$; $f, f', f'', f^{(3)}, f^{(4)} \in L_1(\mathbb{R}) \cap C(\mathbb{R})$; $\xi > 0, x \in \mathbb{R}$. Then*

$$\begin{aligned}
\|E_3\|_1 &= \|T_{r,\xi}(f, x) - f(x) - \\
& \left(\frac{2f''(x)}{\beta^2 - \alpha^2} \right) \frac{1}{\lambda} \left[\sum_{j=0}^r \alpha_j \left(\int_0^\infty (\cos(\alpha jt) - \cos(\beta jt)) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) \right] - \\
& \left(\frac{4(f^{(4)}(x) + (\alpha^2 + \beta^2)f''(x))}{(\alpha\beta)^2(\beta^2 - \alpha^2)} \right) \frac{1}{\lambda} \\
& \left[\beta^2 \sum_{j=0}^r \alpha_j \left(\int_0^\infty \sin^2\left(\frac{\alpha jt}{2}\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) - \right. \\
& \left. \alpha^2 \sum_{j=0}^r \alpha_j \left(\int_0^\infty \sin^2\left(\frac{\beta jt}{2}\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) \right] \Bigg\|_1 \leq \\
& \left[\frac{4}{|\beta^2 - \alpha^2| \lambda_1} \right]
\end{aligned} \tag{93}$$

$$\left[\sum_{j=1}^r |\alpha_j| j^2 \left[\psi_{12} + \frac{j}{2} \psi_{23} \right] \right]$$

$$\omega_1 \left(f^{(4)} + (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f, \xi \right)_1 \cdot \xi^2 =: \Psi_3(\xi) \rightarrow 0,$$

as $\xi \rightarrow 0$.

Proof. We have that (F is as in (25))

$$\begin{aligned} \int_{-\infty}^{\infty} |E_3(x)| dx &\stackrel{(82)}{=} \int_{-\infty}^{\infty} \left(\frac{1}{\lambda} \left| \int_{-\infty}^{\infty} R(t) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right| \right) dx \leq \\ &\int_{-\infty}^{\infty} \left(\frac{1}{\lambda} \int_{-\infty}^{\infty} |R(t)| \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) dx \stackrel{(30)}{\leq} \frac{2}{|\beta^2 - \alpha^2| \lambda} \\ &\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left[\sum_{j=1}^r |\alpha_j| j^2 \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)| (|t| - \theta) d\theta \right) \right] \right. \\ &\quad \left. \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right) dx \leq \tag{94} \\ &\frac{2}{|\beta^2 - \alpha^2| \lambda} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left[\sum_{j=1}^r |\alpha_j| j^2 \right. \right. \\ &\quad \left. \left. \left(\int_0^{|t|} |F(x + j \operatorname{sign}(t)\theta) - F(x)| d\theta \right) |t| \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right] \right) dx = \\ &\frac{2}{|\beta^2 - \alpha^2| \lambda} \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(\int_{-\infty}^{\infty} \right. \right. \right. \\ &\quad \left. \left. |F(x + j \operatorname{sign}(t)\theta) - F(x)| dx \right) d\theta \right) |t| \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right] \leq \\ &\frac{2}{|\beta^2 - \alpha^2| \lambda} \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \omega_1 \left(F, \xi \frac{j\theta}{\xi} \right)_1 d\theta \right) |t| \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right] \leq \end{aligned}$$

$$\begin{aligned}
& \frac{2}{|\beta^2 - \alpha^2| \lambda} \omega_1(F, \xi)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(1 + \frac{j\theta}{\xi} \right) d\theta \right) |t| \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right] = \\
& \frac{2}{|\beta^2 - \alpha^2| \lambda} \omega_1(F, \xi)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \int_{-\infty}^{\infty} \left(|t| + \frac{j}{2\xi} t^2 \right) |t| \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right] = \\
& \frac{4}{|\beta^2 - \alpha^2| \lambda} \omega_1(F, \xi)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \int_0^{\infty} \left(t + \frac{j}{2\xi} t^2 \right) t \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right] = \\
& \frac{4}{|\beta^2 - \alpha^2| \lambda} \omega_1(F, \xi)_1 \sum_{j=1}^r |\alpha_j| j^2 \\
& \left[\int_0^{\infty} t^2 \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt + \frac{j}{2\xi} \int_0^{\infty} t^3 \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right] = \\
& \frac{4\xi^{3-2\bar{\beta}}}{|\beta^2 - \alpha^2| \lambda} \omega_1(F, \xi)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \left[\int_0^{\infty} x^2 \left(\frac{\sin x}{x} \right)^{2\bar{\beta}} dx + \right. \right. \\
& \left. \left. \frac{j}{2} \int_0^{\infty} x^3 \left(\frac{\sin x}{x} \right)^{2\bar{\beta}} dx \right] \right] =
\end{aligned} \tag{95}$$

(we use [11], p. 210, formula 1033)

$$\frac{4\xi^2}{|\beta^2 - \alpha^2| \lambda_1} \omega_1(F, \xi)_1 \left[\sum_{j=1}^r |\alpha_j| j^2 \left[\psi_{12} + \frac{j}{2} \psi_{23} \right] \right] < \infty. \tag{96}$$

The claim is proved. ■

We finish this work with the following result.

Corollary 12 (to Theorem 11) *It holds*

$$\begin{aligned}
\|T_{r,\xi}(f) - f\|_1 & \leq \Psi_3(\xi) + \|f''\|_1 \left(\frac{2\xi^{2\bar{\beta}-1}}{\lambda_1 |\beta^2 - \alpha^2|} \right) \\
& \left[\sum_{j=0}^r |\alpha_j| \int_0^{\infty} |\cos(\alpha_j t) - \cos(\beta_j t)| \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\bar{\beta}} dt \right] +
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{4 \|f^{(4)}\|_1 + (\alpha^2 + \beta^2) \|f''\|_1}{(\alpha\beta)^2 |\beta^2 - \alpha^2|} \right) \frac{\xi^{2\bar{\beta}-1}}{\lambda_1} \tag{97} \\
& \left[\beta^2 \sum_{j=0}^r |\alpha_j| \left(\int_0^\infty \sin^2 \left(\frac{\alpha jt}{2} \right) \left(\frac{\sin \left(\frac{t}{\xi} \right)}{t} \right)^{2\bar{\beta}} dt \right) + \right. \\
& \left. \alpha^2 \sum_{j=0}^r |\alpha_j| \left(\int_0^\infty \sin^2 \left(\frac{\beta jt}{2} \right) \left(\frac{\sin \left(\frac{t}{\xi} \right)}{t} \right)^{2\bar{\beta}} dt \right) \right] \rightarrow 0,
\end{aligned}$$

as $\xi \rightarrow 0$. Above $\Psi_3(\xi)$ is as in (93).

Proof. As similar to Corollary 10 is omitted. ■

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