THE NEW UPPER LIMITS FOR H-H INTEGRAL INEQUALITY

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ABSTRACT. In this study, new upper bounds for the right and left sides of Hermite- Hadamard (in short HH) integral inequality were obtained by writing 4 new lemmas for differentiable mappings. To do this, we used some classical inequalities. And it turns out that the Power Mean Integral inequality is more suitable for capturing the smallest upper bound.

1. INTRODUCTION

We need some representations and preliminary information that appear in the inequalities we will write. The following definitions are well known in the literature:

Definition 1. *The function* $f : [a, b] \to \mathbb{R}$, *is said to be convex, if we have*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

for all $x, y \in [a, b]$ *and* $t \in [0, 1]$.

Geometrically, this means that if A, B and $C$are three distinct points on the graph of f with B between A and C, then B is on or below chord AC. A huge amount of the researchers interested in this definition and there are several papers based on convexity. Many important inequalities are established for the class of convex functions, but one of the most important is so called Hermite-Hadamard's inequality (or Hadamard's inequality). This double inequality well known in the literature is stated as follows

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and let $a, b \in I$, with $a < b$. The following double inequality (HH in short)

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

is known as Hermite- Hadamard's inequality for convex functions. The above inequality is in the reversed direction if f is concave. Kırmacı obtained several upper bounds on the left side of the (1.1) integral inequality for differentiable convex mappings in [1].

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Definition 2. *Let real function f be defined on some nonempty interval I of real numbers line R. The function f is said to be quasi-convex on I if inequality*

$$f(tx+(1-t)y) \leq \sup\{f(x), \ f(y)\} \qquad (QC)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Clearly, any convex function is a quasi-convex function but every quasi-convex function is not convex function.

For example the function $f : R^{+} \to R, \ f(x) = \ln x, \ x \in R^{+}$ is quasi-convex but it is not convex.

In order to write better upper bounds, we will also make use of classical inequalities such as Hölder and Power mean, which are well known in the literature:

Hölder integral inequality : Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on $[a, b]$ and if $|f|^p$ and $|g|^q$ are integrable functions on $[a, b]$, then

$$\int_a^b |f(x)g(x)|\,dx \leq \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\,dx\right)^{\frac{1}{q}} \tag{1.2}$$

If with equality holding if and only if $K|f(x)|^p = L|g(x)|^q$ almost everywhere, where K and L are constants. This study is another version of the study in

Power mean integral inequalitiy : Let $q \geq 1$ and $\mathrm{p}^{-1} + q^{-1} = 1$,If f an d g real functions defined on $[a, b]$ and $|f|$ and $|f|, \ |f|.|g|^q$ are integrable functions on $[a, b]$.Then

$$\int_a^b |f(x)g(x)|\,dx \leq \left(\int_a^b |f(x)|\,dx\right)^{1-\frac{1}{q}} \left(\int_a^b |f(x)| \ |g(x)|^q\,dx\right)^{\frac{1}{q}} \tag{1.3}$$

In this study, we obtained better upper bounds for both the right and left sides of (1.1) by using new lemmas. In other words, upper limits that were smaller than the obtained upper limits were written.

Theorem 1. *[1]: Let $f : I^* \subset R \to R$ be a differentiable mapping on I^*, $b, a \in I^*, a < b$, If $|f'|$ is convex on $[a, b]$, Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{8}\left(|f'(a)| + |f'(b)|\right). \tag{1.4}$$

Theorem 2. *[1] Let $f : I^* \subset R \to R$ be a differentiable mapping on I^*, $b, a \in I^*, a < b$, and let $p > 1$. If the mapping $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$, Then we have*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{b-a}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(|f'(a)| + |f'(b)|\right)\right]. \end{aligned} \tag{1.5}$$

If we take the limit of both sides of the inequality (1.5) for $p \to \infty$. we get

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq (b-a) \ \left[\left(|f'(a)| + |f'(b)|\right)\right].$$

Alomari et al. interpreted the following inequality for the right side of (1.1) in [2].

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)^2}{12}\max\left\{\left|f''(a)\right|,\left|f''(b)\right|\right\}$$

where $f:I\subset R\to R$ be a twice integrable mapping on $I^0, a,b\in I$ with $a<b$.

We wrote new integral inequalities regarding both the right and left sides of (1.1) in Theorems using different lemmas. Researchers interested in the subject can look at the articles in [3] – [13].

2. The Results

The first two of the lemmas we wrote below are related to the right side of (1.1) and the last two are related to the left side. We will prove our basic results with the help of (1.2)-(1.3) classical inequalities and the lemmas we wrote and proved below.

Lemma 1. *Suppose that* $f:I^*\subset R\to R$ *be a differentiable mapping on* I^* (I^* *is interior of* I) *with* $b<a$. *If* $f'\in L[b,a]$, *and* $t\in[0,1]$ *Then we have*
(2.1)

$$\frac{1}{b-a}\left(f(b)-f(a)\right)+\frac{2}{(b-a)^2}\left(\int_a^{\frac{a+b}{2}}f(x)\,dx-\int_{\frac{a+b}{2}}^{b}f(x)\,dx\right)=\left\{\begin{array}{c}\int_0^{\frac{1}{2}}(1-2t)\,f'(tb+(1-t)a)\,dt\\+\int_{\frac{1}{2}}^{1}(2t-1)\,f'(tb+(1-t)a)\,dt\end{array}\right\}$$

Proof. By integration by parts

$$\begin{aligned}&\int_0^{\frac{1}{2}}(1-2t)\,f'(tb+(1-t)a)\,dt\\=&-\frac{1}{b-a}f(a)+\frac{2}{(b-a)^2}\int_0^{\frac{1}{2}}f(tb+(1-t)a)\,dt\\=&-\frac{1}{b-a}f(a)+\frac{2}{(b-a)^2}\int_a^{\frac{a+b}{2}}f(x)\,dx\end{aligned}$$

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1} (2t-1) f'(tb+(1-t)a)\,dt \\
&= \frac{1}{b-a} f(b) - \frac{2}{(b-a)^2} \int_{\frac{1}{2}}^{1} f(tb+(1-t)a)\,dt \\
&= \frac{1}{b-a} f(b) + \frac{2}{(b-a)^2} \int_{b}^{\frac{a+b}{2}} f(x)\,dx
\end{aligned}$$

□

If the last two equalities are added side by side, the proof is completed.

Lemma 2. *Suppose that $f : I^* \subset R \to R$ be a differentiable mapping on I^* (I^* is interior of I) with $a < b$. If $f' \in L[a,b]$, and $t \in [0,1]$ Then we have*

$$\int_{a}^{b} f(x)\,dx - \frac{(b-a)}{2}[f(a)+f(b)] \tag{2.2}$$

$$\frac{(b-a)^2}{2}\left\{ \int_{0}^{\frac{1}{2}} (1-2t) f'(tb+(1-t)a)\,dt + \int_{\frac{1}{2}}^{1} (1-2t) f'(tb+(1-t)a)\,dt \right\} \tag{2.$\cancel{3}$}$$

Proof. By integration by parts

$$\int_{0}^{\frac{1}{2}} (1-2t) f'(tb+(1-t)a)\,dt \tag{2.4}$$

$$= -\frac{1}{b-a} f(a) + \frac{2}{(b-a)} \int_{0}^{\frac{1}{2}} f(tb+(1-t)a)\,dt \tag{2.5}$$

□

$$= -\frac{1}{b-a} f(a) + \frac{2}{(b-a)^2} \int_{a}^{\frac{a+b}{2}} f(x)\,dx$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1} (1-2t) f'(tb+(1-t)a)\,dt \\
&= \frac{1}{b-a} f(b) + \frac{2}{(b-a)} \int_{\frac{1}{2}}^{1} f(tb+(1-t)a)\,dt
\end{aligned}$$

$$= \frac{1}{b-a} f(b) + \frac{2}{(b-a)^2} \int_{\frac{a+b}{2}}^{b} f(x)\, dx$$

A combination of the last equations above gives the following inequality:

$$\int_{0}^{\frac{1}{2}} (1-2t) f'(tb + (1-t)a)\, dt + \int_{\frac{1}{2}}^{1} (2t-1) f'(tb + (1-t)a)\, dt$$

$$= \frac{1}{b-a} (f(b) - f(a)) + \frac{2}{(b-a)^2} \left\{ \int_{a}^{\frac{a+b}{2}} f(x)\, dx + \int_{\frac{a+b}{2}}^{b} f(x)\, dx \right\}$$

$$= \frac{1}{b-a} (f(b) - f(a)) + \frac{2}{(b-a)^2} \int_{a}^{b} f(x)\, dx$$

which completes the proof. So we get(2.2).

Lemma 3. *Suppose that* $f : I^* \subset R \to R$ *be a differentiable mapping on* I^* $(I^*$ *is interior of* $I)$ *with* $a < b$*. If* $f'' \in L[a,b]$, *and* $t \in [0,1]$ *Then we have*

$$f'\left(\frac{a+b}{2}\right) - \frac{2}{b-a} f(b) + \frac{2}{(b-a)^3} \int_{a}^{b} f(x)\, dx \tag{2.6}$$

$$= \int_{0}^{\frac{1}{2}} t^2 f''(tb + (1-t)a)\, dt + \int_{\frac{1}{2}}^{1} \left(t^2 - 1\right) f''(tb + (1-t)a)\, dt \tag{2.7}$$

Proof. If the partial integration method is applied twice consecutively to each of the integrals in equation(2.6) . We easily obtain the left side of (2.6) . □

Lemma 4. *Suppose that* $f : I^* \subset R \to R$ *be a differentiable mapping on* I^* $(I^*$ *is interior of* $I)$ *with* $a < b$*. If* $f'' \in L[a,b]$, *and* $t \in [0,1]$ *Then we have*

$$\frac{1}{b-a} f'(a) + \frac{2}{(b-a)^3} \int_{a}^{b} f(x)\, dx - f'\left(\frac{a+b}{2}\right) \tag{2.8}$$

$$= \int_{0}^{\frac{1}{2}} \left(t^2 - 1\right) f''(tb + (1-t)a)\, dt + \int_{\frac{1}{2}}^{1} (1-t)^2 f''(tb + (1-t)a)\, dt \tag{2.9}$$

Proof. As in the proof of Lemma 1,If the partial integration method is applied twice consecutively to each of the integrals in equation(2.8) . We easily obtain the left side of (2.8) . □

Theorem 3. *Let $f : I^* \subset R \to R$ be a differentiable mapping on I^*, $b, a \in I^*, b < a$, If $|f'|$ is convex on $[b, a]$, Then we have*

$$\left| \frac{1}{a-b}\left(f(a) - f(b)\right) + \frac{2}{(a-b)^2}\left(\int_a^{\frac{a+b}{2}} f(x)\,dx - \int_{\frac{a+b}{2}}^{b} f(x)\,dx \right) \right| \tag{2.10}$$

$$\leq \frac{|f'(b)|}{4} + \frac{|f'(a)|}{6} \tag{2.11}$$

Proof. First of all, from (2.1) and the properties of modulus

$$\left| \frac{1}{a-b}\left(f(a) - f(b)\right) + \frac{2}{(a-b)^2}\left(\int_a^{\frac{a+b}{2}} f(x)\,dx - \int_{\frac{a+b}{2}}^{b} f(x)\,dx \right) \right| \tag{2.12}$$

$$\leq \int_0^{\frac{1}{2}} (1-2t)\left|f'(tb + (1-t)a)\right| dt + \int_{\frac{1}{2}}^{1} (2t-1)\left|f'(tb + (1-t)a)\right| dt$$

□

Using the convexity of $|f'|$ with Calculation of integrals on the right side of (2.12), follows that

$$\int_0^{\frac{1}{2}} (1-2t)\left|f'(tb + (1-t)a)\right| dt \tag{2.13}$$

$$\leq |f'(b)| \int_0^{\frac{1}{2}} t(1-2t)\,dt + |f'(a)| \int_0^{\frac{1}{2}} (1-2t)(1-t)\,dt \tag{2.14}$$

$$= \frac{|f'(b)|}{24} + \frac{5|f'(a)|}{24}$$

and

$$\left| \int_{\frac{1}{2}}^{1} (2t-1) f'(tb + (1-t)a)\,dt \right| \tag{2.15}$$

$$\leq \int_{\frac{1}{2}}^{1} (2t-1)\left|f'(tb + (1-t)a)\right| dt \leq \tag{2.16}$$

$$\leq |f'(b)| \int_{\frac{1}{2}}^{1} (2t-1)t + |f'(a)| \int_0^{\frac{1}{2}} (2t-1)(1-t)\,dt$$

$$= \frac{5|f'(b)|}{24} - \frac{|f'(a)|}{24}$$

A combination of (2.12)-(2.15) gives the required inequality (2.10).

Theorem 4. *Let $f : I^* \subset R \to R$ be a differentiable mapping on I^*, $a, b \in I^*, a < b$, If $|f'|$ is convex on $[a, b]$, Then we have*

$$\left|\frac{1}{b-a}\left(f(b)-f(a)\right)+\frac{2}{(b-a)^2}\int_a^b f(x)\,dx\right| \tag{2.17}$$

$$\leq \frac{(b-a)^2}{12}\left[\left|f'(a)\right|-\left|f'(b)\right|\right] \tag{2.18}$$

Proof. From the properties of modulus, Using the equality (2.2) and the convexity of $|f'|$,it follows that □

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{(b-a)}{2}\left[f(a)+f(b)\right]\right| \\
&= \frac{(b-a)^2}{2}\left|\int_0^{\frac{1}{2}}(1-2t)\,f'(tb+(1-t)a)\,dt+\int_{\frac{1}{2}}^1(1-2t)\,f'(tb+(1-t)a)\,dt\right| \\
&\leq \int_0^{\frac{1}{2}}\left|(1-2t)\right|\left|f'(tb+(1-t)a)\right|dt+\int_{\frac{1}{2}}^1\left|(1-2t)\right|\left|f'(tb+(1-t)a)\right|dt \\
&= \int_0^{\frac{1}{2}}(1-2t)\left|f'(tb+(1-t)a)\right|dt+\int_{\frac{1}{2}}^1(2t-1)\left|f'(tb+(1-t)a)\right|dt
\end{aligned}$$

The proof is complete when the final integrals are calculated, we get the (2.17) .

Theorem 5. *Let $f : I^* \subset R \to R$ be a differentiable mapping on I^*, $b, a \in I^*, b < a$, If $|f'|^{\frac{p}{p-1}} = |f'|^q$ is convex on $[b, a]$, $p > 1$ .Then following inequalities hold :*

$$\left|\frac{1}{a-b}\left(f(a)-f(b)\right)+\frac{2}{(a-b)^2}\left(\int_a^{\frac{a+b}{2}} f(x)\,dx-\int_{\frac{a+b}{2}}^b f(x)\,dx\right)\right| \tag{2.19}$$

$$\leq \frac{1}{2^{1-\frac{2}{p}}(p+1)^{\frac{1}{p}}}\left(\left|f'(b)\right|+\left|f'(a)\right|\right)$$

Where $p^{-1}+q^{-1}=1$
and

$$\left|\frac{1}{a-b}\left(f(a)-f(b)\right)+\frac{2}{(a-b)^2}\left(\int_a^{\frac{a+b}{2}} f(x)\,dx-\int_{\frac{a+b}{2}}^b f(x)\,dx\right)\right| \tag{2.20}$$

$$\leq \frac{1}{2}\left(\left|f'(b)\right|+\left|f'(a)\right|\right) \tag{2.21}$$

Proof. Using the equality(2.1) and Hölders' integral inequalities with properties of the modulus

$$\left|\frac{1}{b-a}\left(f\left(b\right)-f\left(a\right)\right)+\frac{2}{\left(b-a\right)^{2}}\left(\int_{a}^{\frac{a+b}{2}}f\left(x\right)dx-\int_{\frac{a+b}{2}}^{b}f\left(x\right)dx\right)\right|$$

$$\leq\int_{0}^{\frac{1}{2}}\left(1-2t\right)\left|f'\left(tb+\left(1-t\right)a\right)\right|dt+\int_{\frac{1}{2}}^{1}\left(2t-1\right)\left|f'\left(tb+\left(1-t\right)a\right)\right|dt$$

$$\begin{aligned}\leq\quad&\left(\int_{0}^{\frac{1}{2}}\left(1-2t\right)^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}}\left|f'\left(tb+\left(1-t\right)a\right)\right|^{q}dt\right)^{\frac{1}{q}}\\&+\left(\int_{\frac{1}{2}}^{1}\left(2t-1\right)^{p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|f'\left(tb+\left(1-t\right)a\right)\right|^{q}dt\right)^{\frac{1}{q}}\end{aligned}$$

□

Finally, if we use the convexity of $|f'|^{\frac{p}{p-1}}=.\ |f'|^{q}$ with the calculation of the necessary integrals we get

$$\left(\int_{0}^{\frac{1}{2}}\left(1-2t\right)^{p}dt\right)^{\frac{1}{p}}=\frac{1}{2^{\frac{1}{p}}\left(p+1\right)^{\frac{1}{p}}}=\left(\int_{\frac{1}{2}}^{1}\left(2t-1\right)^{p}dt\right)^{\frac{1}{p}}$$

$$\int_{0}^{\frac{1}{2}}\left|f'\left(tb+\left(1-t\right)a\right)\right|^{q}dt\leq\frac{\left|f'\left(b\right)\right|^{q}+3\left|f'\left(a\right)\right|^{q}}{8}$$

$$\int_{\frac{1}{2}}^{1}\left|f'\left(tb+\left(1-t\right)a\right)\right|^{q}dt\leq\frac{3\left|f'\left(b\right)\right|^{q}+\left|f'\left(a\right)\right|^{q}}{8}$$

and
When the above integrals are written in place, we get the inequality (2.19).
For the inequality (2.20)
Since $0\leq\frac{p-1}{p}<1,$

$$\sum_{i=1}^{n}\left(a_{i}+b_{i}\right)^{k}\leq\sum_{i=1}^{n}a_{i}^{k}+\sum_{i=1}^{n}b_{i}^{k}\tag{2.22}$$

for $(0\leq k<1,\ )$, $a_i,b_i\geq 0$ $(i=1,2,...)$ we obtain

$$\left|\frac{1}{a-b}\left(f\left(a\right)-f\left(b\right)\right)+\frac{2}{\left(a-b\right)^{2}}\left(\int_{a}^{\frac{a+b}{2}}f\left(x\right)dx-\int_{\frac{a+b}{2}}^{b}f\left(x\right)dx\right)\right| \tag{2.23}$$

$$\leq\frac{1}{2^{1-\frac{2}{p}}\left(p+1\right)^{\frac{1}{p}}}\left(\left|f'\left(b\right)\right|+\left|f'\left(a\right)\right|\right) \tag{2.24}$$

Since

$$\frac{1}{2}<\frac{1}{2^{1-\frac{2}{p}}\left(p+1\right)^{\frac{1}{p}}}<1$$

and finally, If we take the limit of both sides of (2.23) for $p\rightarrow\infty$ we get inequality (2.20).

Here we obtained two new upper bounds for the left side of (2.19). Researchers on the subject can write new upper bounds, each time using different types of convexity.

Theorem 6. *Let $f:I\subset R\rightarrow R$ be a differentiable mapping on $I^{*},b,a\in I$ with $b<a$ an f' is integrable on $[b,a]$. If $|f'|^{q}$ is an convex on $[b,a]$, $q\geq 1$, $t\in[0,1]$. Then the following inequality holds:*

$$\left|\frac{1}{a-b}\left(f\left(a\right)-f\left(b\right)\right)+\frac{2}{\left(a-b\right)^{2}}\left(\int_{a}^{\frac{a+b}{2}}f\left(x\right)dx-\int_{\frac{a+b}{2}}^{b}f\left(x\right)dx\right)\right| \tag{2.25}$$

$$\leq\left(\frac{1}{4}\right)^{2-\frac{1}{q}}\left[\left|f'\left(b\right)\right|+\frac{11}{6}\left|f'\left(a\right)\right|\right]$$

Proof. From the properties of Modulus, (2.1), the well known power mean inequality and convexity of $|f'|^{q}$, respectively, we have

$$\left|\frac{1}{a-b}\left(f\left(a\right)-f\left(b\right)\right)+\frac{2}{\left(a-b\right)^{2}}\left(\int_{a}^{\frac{a+b}{2}}f\left(x\right)dx-\int_{\frac{a+b}{2}}^{b}f\left(x\right)dx\right)\right|$$

$$=\left|\int_{0}^{\frac{1}{2}}\left(1-2t\right)f'\left(tb+\left(1-t\right)a\right)dt+\int_{\frac{1}{2}}^{1}\left(2t-1\right)f'\left(tb+\left(1-t\right)a\right)dt\right|$$

$$\leq\int_{0}^{\frac{1}{2}}\left(1-2t\right)\left|f'\left(tb+\left(1-t\right)a\right)\right|dt+\int_{\frac{1}{2}}^{1}\left(2t-1\right)\left|f'\left(tb+\left(1-t\right)a\right)\right|dt$$

$$\leq\left(\int_{0}^{\frac{1}{2}}\left(1-2t\right)dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{1}{2}}\left(1-2t\right)\left|f'\left(tb+\left(1-t\right)a\right)\right|^{q}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}(2t-1)\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}(2t-1)\left|f'\left(tb+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}}$$

$$=\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left\{\left[\frac{|f'(b)|^{q}}{24}+\frac{|f'(a)|^{q}}{2}\right]+\frac{5\,|f'(b)|^{q}}{24}-\frac{|f'(a)|^{q}}{24}\right\}$$

$$=\left(\frac{1}{4}\right)^{2-\frac{1}{q}}\left\{|f'(b)|+\frac{11}{6}|f'(a)|\right\}$$

□

which completes the required proof .

Here we used the facts

$$\left(\int_{0}^{\frac{1}{2}}(1-2t)\,dt\right)^{1-\frac{1}{q}}=\left(\int_{\frac{1}{2}}^{1}(2t-1)\,dt\right)^{1-\frac{1}{q}}=\left(\frac{1}{4}\right)^{1-\frac{1}{q}}$$

$$\left(\int_{\frac{1}{2}}^{1}(2t-1)\left|f'\left(tb+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}}=\left(\frac{5\,|f'(b)|^{q}}{24}-\frac{|f'(a)|^{q}}{24}\right)^{\frac{1}{q}}$$

$$\sum_{i=1}^{n}(a_i+b_i)^{k}\leq\sum_{i=1}^{n}a_i^{k}+\sum_{i=1}^{n}b_i^{k},\ \ (0\leq k<1,\ ),\ a_i,b_i\geq 0\ \ (i=1,2,...)$$

Corollary 1. *Since the left sides of* (2.20) *and* (2.25)*coincide, we have*

$$\left|\frac{1}{a-b}\left(f(a)-f(b)\right)+\frac{2}{(a-b)^{2}}\left(\int_{a}^{\frac{a+b}{2}}f(x)\,dx-\int_{\frac{a+b}{2}}^{b}f(x)\,dx\right)\right|$$

$$\leq\min\left(\frac{|f'(a)|+|f'(b)|}{2},\ \left(\frac{1}{4}\right)^{2-\frac{1}{q}}\left\{|f'(b)|+\frac{11}{6}|f'(a)|\right\}\right)$$

On the other hand, Since

$$\frac{1}{4}\leq\left(\frac{1}{4}\right)^{2-\frac{1}{q}}\left\{|f'(b)|+\frac{11}{6}|f'(a)|\right\}<\frac{1}{16}$$

Then

$$\left|\frac{1}{a-b}\left(f(a)-f(b)\right)+\frac{2}{(a-b)^{2}}\left(\int_{a}^{\frac{a+b}{2}}f(x)\,dx-\int_{\frac{a+b}{2}}^{b}f(x)\,dx\right)\right|$$

$$\leq\min\left(\frac{|f'(a)|+|f'(b)|}{2},\ \frac{|f'(b)|+\frac{11}{6}|f'(a)|}{16}\right)=\frac{|f'(b)|+\frac{11}{6}|f'(a)|}{16}$$

That is, $L = \frac{\left|f'(b)\right| + \frac{11}{6}\left|f'(a)\right|}{16}$ number is a better upper bound for

$$K = \left| \frac{1}{a-b} \left(f(a) - f(b) \right) + \frac{2}{(a-b)^2} \left(\int_a^{\frac{a+b}{2}} f(x)\,dx - \int_{\frac{a+b}{2}}^{b} f(x)\,dx \right) \right|$$

In other words, L is closer to Eküs(K).

The inequalities (2.26) and(2.28), which we wrote and proved below, are the upper bounds on the left and right sides of (1.1) respectively.

Theorem 7. *Let $f : I^* \subset R \to R$ be a differentiable mapping on I^*, $a, b \in I^*, a < b$, If $|f''|$ −quasi convex is convex on $[a, b]$, Then we have*

$$\left| \frac{1}{b-a} f'(a) + \frac{2}{(b-a)^3} \int_a^b f(x)\,dx - f'\left(\frac{a+b}{2}\right) \right| \le \frac{\sup\left\{ |(f''a)|, |f''(b)| \right\}}{12}. \tag{2.26}$$

Proof. From the equality (2.8) and using the convexity of $|f''|$ with the properties of modulus, it follows that

$$\begin{aligned}
&\left| \int_0^{\frac{1}{2}} \left(t^2 - 1\right) f''(tb + (1-t)a)\,dt + \int_{\frac{1}{2}}^{1} (1-t)^2 f''(tb + (1-t)a)\,dt \right| \\
&\int_0^{\frac{1}{2}} \left|\left(t^2 - 1\right)\right| \left|f''(tb + (1-t)a)\right| dt + \int_{\frac{1}{2}}^{1} \left|(1-t)^2\right| \left|f''(tb + (1-t)a)\right| dt \\
\le\ & \sup\left\{ |(f''a)|, |f''(b)| \right\} \left\{ \int_0^{\frac{1}{2}} \left|\left(t^2 - 1\right)\right| dt + \int_{\frac{1}{2}}^{1} \left|(1-t)^2\right| dt \right\} \\
=\ & \sup |(f''a)|, |f''(b)| \left\{ \int_0^{\frac{1}{2}} \left(1 - t^2\right) dt + \int_{\frac{1}{2}}^{1} (1-t)^2\,dt \right\} \\
=\ & \sup |(f''a)|, |f''(b)| \left\{ \frac{1}{12} \right\}
\end{aligned}$$

□

which the completed the proof.

Theorem 8. *Suppose that $f : I^* \subset R \to R$ be a differentiable mapping on I^* (I^* is interior of I) with $a < b$. If $|f''|^q$ −quasi convex is convex on $[a, b]$, $q > 1$ Then we have*

(2.27)

$$\left| \frac{1}{b-a} f'(a) + \frac{2}{(b-a)^3} \int_a^b f(x)\,dx - f'\left(\frac{a+b}{2}\right) \right| \le \frac{2}{3} \left\{ \sup\left(|f''(b)|^q, |f''(a)|^q \right) \right\}^{\frac{1}{q}}$$

where $p^{-1}+q^{-1}=1$,

Proof. From (2.8), properties of modulus, power mean inequality and $|f''|^q$ −quasi convex, respectively. we have

$$\begin{aligned}
&\left|\frac{1}{b-a}f'(a)+\frac{2}{(b-a)^3}\int_a^b f(x)\,dx-f'\left(\frac{a+b}{2}\right)\right| \\
&= \left|\int_0^{\frac{1}{2}}\left(t^2-1\right)f''(tb+(1-t)a)\,dt+\int_0^{\frac{1}{2}}(1-t)^2 f''(tb+(1-t)a)\,dt\right| \\
&\leq \int_0^{\frac{1}{2}}\left|\left(t^2-1\right)\right|\ \left|f''(tb+(1-t)a)\right|dt+\int_{\frac{1}{2}}^1\left|(1-t)^2\right|\ \left|f''(tb+(1-t)a)\right|dt \\
&\int_0^{\frac{1}{2}}\left(1-t^2\right)\ \left|f''(tb+(1-t)a)\right|dt+\int_{\frac{1}{2}}^1(1-t)^2\ \left|f''(tb+(1-t)a)\right|dt \\
&\left(\int_0^{\frac{1}{2}}\left(1-t^2\right)dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(1-t^2\right)\ \left|f''(tb+(1-t)a)\right|^q\ dt\right)^{\frac{1}{q}} \\
&+\left(\int_{\frac{1}{2}}^1(1-t)^2\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1(1-t)^2\left|f''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \left(\frac{11}{24}\right)^{\frac{1}{p}}\left\{\sup\left\{\left|f''(b)\right|^q,\ \left|f''(a)\right|^q\right\}\right\}^{\frac{1}{q}}\left(\frac{11}{24}\right)^{\frac{1}{q}} \\
&+\left(\frac{5}{24}\right)^{\frac{1}{p}}\left\{\sup\left\{\left|f''(b)\right|^q,\ \left|f''(a)\right|^q\right\}\right\}^{\frac{1}{q}}\left(\frac{5}{24}\right)^{\frac{1}{q}} \\
&= \frac{2\left\{\sup\left\{\left|f''(b)\right|^q,\ \left|f''(a)\right|^q\right\}\right\}^{\frac{1}{q}}}{3}
\end{aligned}$$

□

which completes the required proof.
From (2.26) and (2.27), we obtain

$$\begin{aligned}
&\left|\frac{1}{b-a}f'(a)+\frac{2}{(b-a)^3}\int_a^b f(x)\,dx-f'\left(\frac{a+b}{2}\right)\right| \\
&\leq \min\left\{\frac{2}{3}\left\{\sup\left(\left|f''(b)\right|^q,\left|f''(a)\right|^q\right)\right\}^{\frac{1}{q}},\ \frac{\sup\left\{\left|(f''a)\right|,\left|f''(b)\right|\right\}}{12}\right\}
\end{aligned}$$

Theorem 9. *Suppose that $f : I^* \subset R \to R$ be a differentiable mapping on I^* (I^* is interior of I) with $a < b$. If $|f'|^q$ −quasi convex is convex on $[a,b]$, $q > 1$ Then we have*

(2.28)
$$\left|\int_a^b f(x)\,dx - \frac{(b-a)}{2}\left[f(a)+f(b)\right]\right| \le \frac{(b-a)^2}{(p+1)^{\frac{1}{p}}}\left\{\sup\left(|f'(b)|^q, |f'(a)|^q\right)\right\}^{\frac{1}{q}}$$

where $p^{-1}+q^{-1}=1$,

Proof. From (2.2), properties of modulus, the convexity of $|f'|^q$ and Hölder's inequality, it follows that

$$\frac{(b-a)^2}{2}\left|\left\{\int_0^{\frac{1}{2}}(1-2t)\left|f'(tb+(1-t)a)\right|dt + \int_{\frac{1}{2}}^1 (1-2t)\left|f'(tb+(1-t)a)\right|dt\right\}\right|$$

$$\le \frac{(b-a)^2}{2}\left\{\begin{array}{c}\left(\int_0^{\frac{1}{2}}(1-2t)^p\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}+\\ \left(\int_{\frac{1}{2}}^1(2t-1)^p\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\end{array}\right\}$$

$$\le \frac{(b-a)^2}{2}\left\{\frac{1}{2^{\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[\frac{\left\{\sup\left(|f'(b)|^q,|f'(a)|^q\right)\right\}^{\frac{1}{q}}}{2^{\frac{1}{q}}}+\frac{\left\{\sup\left(|f'(b)|^q,|f'(a)|^q\right)\right\}^{\frac{1}{q}}}{2^{\frac{1}{q}}}\right]\right\}$$

□

$$= \frac{(b-a)^2}{2}\frac{1}{2^{\frac{1}{p}}(p+1)^{\frac{1}{p}}}\frac{1}{2^{\frac{1}{q}}}\left\{2.\left\{\sup\left(|f'(b)|^q,|f'(a)|^q\right)\right\}^{\frac{1}{q}}\right\}$$

$$= \frac{(b-a)^2}{(p+1)^{\frac{1}{p}}}\left\{\sup\left(|f'(b)|^q,|f'(a)|^q\right)\right\}^{\frac{1}{q}}$$

which completes the proof. Here we used the facts that

$$2^{\frac{1}{p}}2^{\frac{1}{q}} = 2^{\frac{1}{p+}+\frac{1}{q}=1} = 2$$

$$\left(\int_0^{\frac{1}{2}}(1-2t)^p\,dt\right)^{\frac{1}{p}} = \frac{1}{2^{\frac{1}{p}}(p+1)^{\frac{1}{p}}} = \left(\int_{\frac{1}{2}}^1(2t-1)^p\,dt\right)^{\frac{1}{p}}$$

$$\left(\int_0^{\frac{1}{2}}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} = \left(\int_{\frac{1}{2}}^1\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} = \frac{\left\{\sup\left(|f'(b)|^q,|f'(a)|^q\right)\right\}^{\frac{1}{q}}}{2^{\frac{1}{q}}}$$

3. Applications

We consider the means for arbitrary real numbers $(b \neq a)$
Arithmetic mean:

$$A(b,a) = \frac{b+a}{2} \quad (b,a \in R)$$

Logarithmic mean:

$$L(b,a) = \frac{b-a}{\ln|b| - \ln|a|} \quad (b \neq a, \quad |b| \neq |a|, \quad b,a \in R)$$

Harmonic mean :

$$H(b,a) = \frac{2}{\frac{1}{b} + \frac{1}{a}}, \quad (b,a \neq 0; \; b,a \in R)$$

Geometric mean :

$$G(b,a) = \sqrt{ab} \quad (ab > 0, b,a \in R)$$

Generalized mean :

$$L_n(b,a) = \left(\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}\right)^{\frac{1}{n}} \quad (n \in Z - \{-1,0\} \;, b \neq a; b,a \in R)$$

Proposition 1. *Let* $b,a \in R$, $b < a$ *and* $n \in N$; $n \geq 2$. *Then we have*

$$\left| nL_{n-1}^{n-1}(a,b) + \frac{4\left[A^{n+1}(a,b) - A\left(a^{n+1}, b^{n+1}\right)\right]}{(a-b)^2(n+1)} - \right|$$
$$\leq \quad n\left(\frac{|b|^{n-1}}{4} + \frac{|a|^{n-1}}{6}\right)$$

Proof. The assertion follows from Theorem 3.; applied for $f(x) = x^n \, (n \geq 2, \; x \in R), a \neq b$ □

Proposition 2. *Let* $b,a \in R \backslash \{0\}$, $a < b$ *and* $n \in N, n \neq \{1,2\}$. *Then we have*

$$\left| nL_{n-1}^{n-1}(b,a) + \frac{2}{(b-a)} L_n^n(b,a) \right| \leq \frac{n(n-1)(b-a)^2}{12} (|a| - |b|) L_{n-2}^{n-2}(|a|, |b|)$$

Proof. The assertion follows from Theorem 4.; applied for $f(x) = x^n \, (n \in N, \; x \in R)$ □

Proposition 3. $a,b \neq 0$, $a,b \in R$, $a < b$.*Then we have*

$$\left| \frac{H^{-1}(-b,a)}{(a-b)} + \frac{1}{(a-b)^2} \ln \frac{A^2(a,b)}{b^2} \right| \leq \frac{A\left(a^{-2}, b^{-2}\right)}{2}$$

Proof. The assertion follows from inequality (2.20) .; applied for $f(x) = \frac{1}{x}, x \neq 0$. □

Proposition 4. $0 \notin [a,b]$, $a < b.n \in N, n \geq 2$. *Then we have*

$$\left| nL_{n-1}^{n-1}(b,a) + \frac{2}{(b-a)} L_n^n(b,a) \right| \leq \frac{1}{16}\left(|b|^{-1} + \frac{11}{6}|a|^{-1}\right)$$

Proof. If we take the limit for $p \to \infty$ in Theorem 6, the assertion follows from inequality (2.25) .; applied for $f(x) = \frac{1}{x}, x \neq 0$. □

Proposition 5. *Let $a, b \in R^{+}$, $a < b$.*

$$\left| \frac{1}{a(b-a)} - \frac{2}{(b-a)^2} + \frac{2}{(b-a)^3} \ln \frac{b^b}{a^a} \right| \leq \sup \left\{ \frac{1}{a^2}, \frac{1}{b^2} \right\} \tag{3.1}$$

Proof. Since every convex function is a quasi-convex the function $f(x) = \frac{1}{x}$ is quasi-convex $(x > 0)$ $(f''(x) > 0)$ if we choose the function $f(x) = \frac{1}{x}$, $(x > 0)$ in Theorem 7, We obtain the following relation.

$$\left| \frac{-1}{a^2(b-a)} + \frac{2}{(b-a)^3} \ln \left(\frac{b}{a} \right) - A^{-2}(a,b) \right| \leq \frac{1}{6} \sup \left\{ a^{-3}, b^{-3} \right\}$$

□

4. Conclusion

Researchers interested in the subject can write new theorems and hence better upper bounds under new lemmas and classical inequalities using different types of convex. Forexample : Every number greater than $K = |f'(b)| + |f'(a)|$ is one of the upper bounds of (2.19). In that case, the supremum of (2.19) is $K = |f'(b)| + |f'(a)|$.

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