Certain refinements of Jordan-type inequalities

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Abstract

We present new sharp bounds for the function $(\sin x)/x$, thus refining the well-known Jordan-type inequalities in the literature. A polynomial-trigonometric approach is used to establish the bounds. The main results are based on the series expansions, monotonicity rules, and the bounds of the ratio of even indexed Bernoulli numbers.

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1. Introduction

Jordan's inequality [1, 3, 17, 23, 27, 34]

$$\frac{2}{\pi} \leq \frac{\sin x}{x} < 1; \;\; x \in (0, \pi/2] \tag{1.1}$$

gives bounds for the sinc function which is defined by $\operatorname{sinc} x = (\sin x)/x$, if $x \neq 0$ and $\operatorname{sinc} x = 1$, if $x = 0$. The double inequality (1.1) is a consequence of the monotonicity of the curve $y = \operatorname{sinc} x$ or the concavity of the curve $y = \sin x$ in $(0, \pi/2]$. It can also be easily achieved through the geometry of circles [32]. Jordan's inequality has its glory among the trigonometric functions due to its importance in calculus and analysis. This much-appreciated inequality motivated many researchers to obtain its refinements, extensions, and generalizations, see for instance [2, 6, 7, 9–12, 15, 16, 18–22, 24–26, 29–31, 33, 35] and the references therein. In 2010, Klén et. al. [17] rediscovered the inequality (1.1) as follows:

$$1 - \frac{x^2}{6} < \frac{\sin x}{x} < 1 - \frac{2x^2}{3\pi^2}; \;\; x \in (-\pi/2, \pi/2). \tag{1.2}$$

Further, in 2022, Bagul and Panchal [8] improved inequality (1.2) to

$$1 - \frac{x^2}{6} < \frac{\sin x}{x} < 1 - \frac{4x^2}{3\pi^2}; \;\; x \in (-\pi/2, \pi/2). \tag{1.3}$$

Recently, the Jordan-type inequalities (1.2) and (1.3) have been generalized and explored in detail in [7]. This article aims to contribute to the field by refining the lower and upper bounds of (1.3). It is worth noting that due to the symmetry of the curves involved, it suffices to improve the inequality (1.3) in $(0, \alpha)$ rather than in $(-\alpha, \alpha)$. The new sharp bounds are established in terms of bounds of inequality (1.2) and cosine function.

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2. Preliminaries and lemmas

We begin by recalling the following series expansions [13, 1.411]:

$$\frac{x}{\sin x} = 1 + \sum_{k=1}^{\infty} \frac{2(2^{2k-1}-1)}{(2k)!}|B_{2k}|x^{2k}; \;\; |x| < \pi, \tag{2.1}$$

$$\cot x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!}|B_{2k}|x^{2k-1}; \;\; |x| < \pi, \tag{2.2}$$

where $|B_{2k}|$ are the absolute even-indexed Bernoulli numbers.

Differentiating (2.2) and then multiplying by $-x^2$ yields

$$\left(\frac{x}{\sin x}\right)^2 = 1 + \sum_{k=1}^{\infty} \frac{2^{2k}(2k-1)}{(2k)!}|B_{2k}|x^{2k}; \;\; |x| < \pi. \tag{2.3}$$

Also, we will employ the following supplementary results.

Lemma 2.1. *[4, 5](l'Hôpital monotone rule) Suppose p and q are any two real numbers such that $p < q$. Let $f_1(x)$ and $f_2(x)$ be two real-valued functions that are continuous on $[p, q]$ and differentiable on (p, q), and $f_2'(x) \neq 0$, for all $x \in (p, q)$. Let,*

$$i(x) = \frac{f_1(x) - f_1(p)}{f_2(x) - f_2(p)}, \quad j(x) = \frac{f_1(x) - f_1(q)}{f_2(x) - f_2(q)}.$$

Then, the functions $i(x)$ and $j(x)$ are increasing (decreasing) on (p, q) if $f_1'(x)/f_2'(x)$ is increasing (decreasing) on (p, q). The strictness of the monotonicity of $i(x)$ and $j(x)$ depends on the strictness of the monotonicity of $f_1'(x)/f_2'(x)$.

The l'Hôpital monotone rule in Lemma 2.1 has been proven to be an important tool in the field of inequalities. The next lemma concerns the monotonicity of the ratio of two series and can be found in [14].

Lemma 2.2. *[14] Let $P(x) = \sum_{k=0}^{\infty} p_k x^k$ and $Q(x) = \sum_{k=0}^{\infty} q_k x^k$ be any two real series converging on the interval $(-R, R)$, where $R > 0$ and $q_k > 0$ for all k. Then the function $P(x)/Q(x)$ is increasing (decreasing) on $(0, R)$ if the sequence $\{p_k/q_k\}$ is increasing (decreasing).*

In addition, we need a double inequality for the ratio of consecutive absolute Bernoulli numbers recently established by Qi [27].

Lemma 2.3. *([27]) For $k \in \mathbb{N}$, the Bernoulli numbers satisfy*

$$\frac{(2^{2k-1}-1)}{(2^{2k+1}-1)} \frac{(2k+1)(2k+2)}{\pi^2} < \frac{|B_{2k+2}|}{|B_{2k}|} < \frac{(2^{2k}-1)}{(2^{2k+2}-1)} \frac{(2k+1)(2k+2)}{\pi^2}.$$

3. Main results

We are now in a position to assert our results with their proofs. First, we use the functions $\left(1 - \frac{x^2}{6}\right)$ and $(1 - \cos x)^2$ to refine inequalities (1.3).

Theorem 3.1. *The function $\left[\frac{\sin x}{x} - \left(1 - \frac{x^2}{6}\right)\right] / (1 - \cos x)^2$ is strictly increasing from $(0, \pi)$ onto (λ_1, λ_2), where $\lambda_1 = \frac{1}{30}$ and $\lambda_2 = \frac{1}{4}\left(\frac{\pi^2}{6} - 1\right)$. In particular,*

- *If $x \in (0, \pi/2)$, then the inequality*

$$\left(1 - \frac{x^2}{6}\right) + \frac{1}{30}(1 - \cos x)^2 < \frac{\sin x}{x} < \left(1 - \frac{x^2}{6}\right) + \left(\frac{\pi^2}{24} + \frac{2}{\pi} - 1\right)(1 - \cos x)^2 \tag{3.1}$$

holds with the optimal constants $\frac{1}{30}$ and $\left(\frac{\pi^2}{24} + \frac{2}{\pi} - 1\right)$ respectively.

- *If $x \in (0,\pi)$, then the inequality*

$$\left(1-\frac{x^2}{6}\right)+\frac{1}{30}(1-\cos x)^2<\frac{\sin x}{x}<\left(1-\frac{x^2}{6}\right)+\frac{1}{4}\left(\frac{\pi^2}{6}-1\right)(1-\cos x)^2 \tag{3.2}$$

holds with the optimal constants $\frac{1}{30}$ and $\frac{1}{4}\left(\frac{\pi^2}{6}-1\right)$ respectively.

Proof. Let us set

$$f(x)=\frac{\left[\frac{\sin x}{x}-\left(1-\frac{x^2}{6}\right)\right]}{(1-\cos x)^2}=\frac{f_1(x)}{f_2(x)};\ \ x\in(0,\pi),$$

where $f_1(x)=\frac{\sin x}{x}-\left(1-\frac{x^2}{6}\right)$ and $f_2(x)=(1-\cos x)^2$ satisfying $f_1(0+)=0=f_2(0)$. After performing the differentiation task, we find

$$\frac{f_1'(x)}{f_2'(x)}=\frac{1}{6}\cdot\frac{3x\cos x-3\sin x+x^3}{x^2\sin x-x^2\sin x\cos x}=\frac{1}{6}\cdot\frac{f_3(x)}{f_4(x)},$$

where $f_3(x)=3x\cos x-3\sin x+x^3$ and $f_4(x)=x^2\sin x-x^2\sin x\cos x$ satisfying $f_3(0)=0=f_4(0)$. Differentiating one more time to apply Lemma 2.1, we get

$$\frac{f_3'(x)}{f_4'(x)}=\frac{1}{2}\frac{x^2\operatorname{cosec}^2 x-x\operatorname{cosec} x}{2x\operatorname{cosec} x-2x\cot x+x^2}.$$

Then utilizing series expansions (2.1)–(2.2), we write

$$\begin{aligned}\frac{f_3'(x)}{f_4'(x)}&=\frac{1}{2}\cdot\frac{1+\sum_{k=1}^{\infty}\frac{2^{2k}(2k-1)}{(2k)!}|B_{2k}|x^{2k}-1-\sum_{k=1}^{\infty}\frac{2(2^{2k-1}-1)}{(2k)!}|B_{2k}|x^{2k}}{2+\sum_{k=1}^{\infty}\frac{2^2(2^{2k-1}-1)}{(2k)!}|B_{2k}|x^{2k}-2+\sum_{k=1}^{\infty}\frac{2^{2k+1}}{(2k)!}|B_{2k}|x^{2k}+x^2}\\&=\frac{1}{2}\cdot\frac{\sum_{k=1}^{\infty}\frac{2}{(2k)!}\left[2^{2k}(2k-1)-(2^{2k-1}-1)\right]|B_{2k}|x^{2k}}{x^2+\sum_{k=1}^{\infty}\frac{2}{(2k)!}\left[2^{2k+1}-2\right]|B_{2k}|x^{2k}}:=\frac{1}{2}\cdot\frac{Q(x)}{x^2+P(x)},\end{aligned}$$

where

$$P(x)=\sum_{k=1}^{\infty}\frac{4}{(2k)!}(2^{2k}-1)|B_{2k}|x^{2k}:=\sum_{k=1}^{\infty}p_k x^{2k}$$

and

$$Q(x)=\sum_{k=1}^{\infty}\frac{2}{(2k)!}\left[2^{2k}(2k-1)-(2^{2k-1}-1)\right]|B_{2k}|x^{2k}:=\sum_{k=1}^{\infty}q_k x^{2k},$$

Here, $q_k>0$ implies that $x^2/Q(x)$ is strictly decreasing. Next we prove that $P(x)/Q(x)$ is also decreasing. So consider

$$\frac{p_k}{q_k}=\frac{2(2^{2k}-1)}{2^{2k}(2k-1)-(2^{2k-1}-1)}:=t_k\ \ \text{(say)}.$$

Now it suffices to show that $t_k>t_{k+1}$, i.e.,

$$\frac{(2^{2k+1}-2)}{2^{2k}(2k-1)-(2^{2k-1}-1)}>\frac{(2^{2k+3}-2)}{2^{2k+2}(2k+1)-(2^{2k+1}-1)}$$

or

$$(2^{2k}-1)(k\cdot 2^{2k+3}+2^{2k+2}-2^{2k+1}+1)>(2^{2k+2}-1)(k\cdot 2^{2k+1}-2^{2k}-2^{2k-1}+1).$$

After simplifying, we get

$$2^{4k+2}-k\cdot 2^{2k+3}-2^{2k+2}+2^{2k+1}>-2^{4k+2}+2^{2k+2}-k\cdot 2^{2k+1}+2^{2k-1},$$

i.e.,

$$16\cdot 2^{4k}+4(k+1)\cdot 2^{2k}>16k\cdot 2^{2k}+16\cdot 2^{2k}+2^{2k}.$$

Equivalently, $2^{2k+4} > 12k+3$, which is true for $k = 1, 2, 3, \cdots$. Thus the sequence $\{p_k/q_k\}$ is decreasing which implies that $P(x)/Q(x)$ is decreasing due to Lemma 2.2. Hence, $Q(x)/(x^2 + P(x))$ is strictly increasing, and by l'Hôpital monotone rule (Lemma 2.1), the function $f(x)$ is also strictly increasing in $(0, \pi)$. As a result, for $x \in (0, \pi/2)$ we have

$$\frac{1}{30} = \lim_{x \to 0^+} f(x) < f(x) < \lim_{x \to \pi/2^-} f(x) = \frac{\pi^2}{24} + \frac{2}{\pi} - 1.$$

This gives inequality (3.1). Similarly for $x \in (0, \pi)$, we have

$$\frac{1}{30} = \lim_{x \to 0^+} f(x) < f(x) < \lim_{x \to \pi^-} f(x) = \frac{1}{4}\left(\frac{\pi^2}{6} - 1\right).$$

This gives inequality (3.2). The proof of Theorem 3.1 is completed. □

Obviously, the lower bound of $(\sin x)/x$ in (1.3) is refined in (3.1). Now suppose

$$U_1(x) = 1 - \frac{4x^2}{3\pi^2} \quad \text{and} \quad U_2(x) = \left(1 - \frac{x^2}{6}\right) + \left(\frac{\pi^2}{24} + \frac{2}{\pi} - 1\right)(1 - \cos x)^2.$$

Then the following Figure 1 shows that the upper bound of $(\sin x)/x$ in (3.1) is also sharper than the corresponding upper bound in (1.3). The curves $(\sin x)/x$ and $U_2(x)$ are almost confounded.

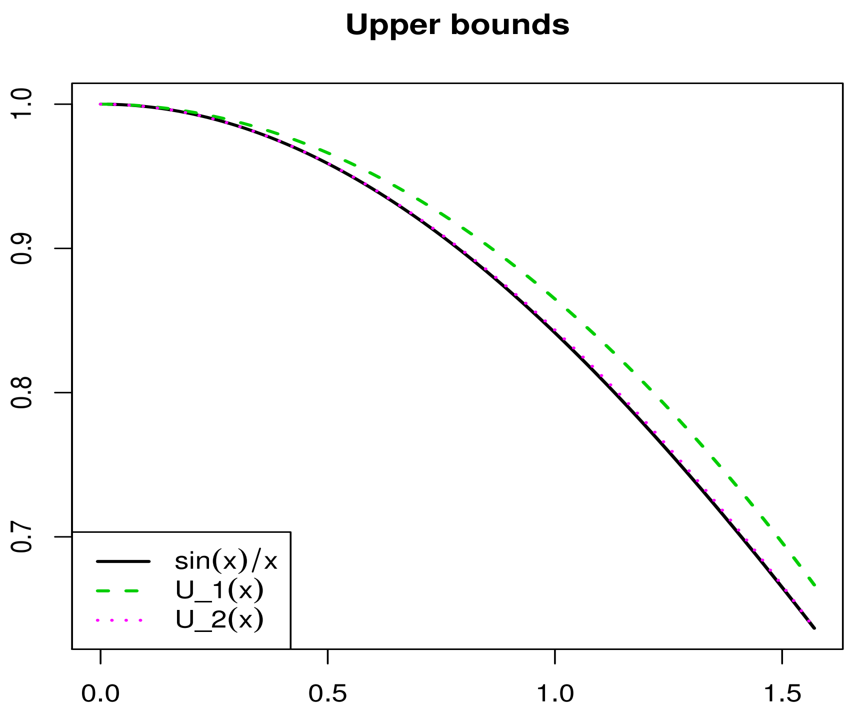

Figure 1. Graphs of upper bounds of $\sin(x)/x$ in (1.3) and (3.1) for $x \in (0, \pi/2)$

Moreover, in (3.2), we obtained the bounds of $(\sin x)/x$ in a wider range of values of x, i.e., in $(0, \pi)$.

In the next theorem, we use the functions $1 - \frac{2x^2}{3\pi^2}$ and $(1 - \cos x)$ to refine inequalities (1.3).

Theorem 3.2. *The function* $\left[\frac{\sin x}{x} - \left(1 - \frac{2x^2}{3\pi^2}\right)\right]/(1-\cos x)$ *is strictly increasing from* $(0,\pi)$ *onto* (δ_1, δ_2)*, where* $\delta_1 = 2\left(\frac{2}{3\pi^2} - \frac{1}{6}\right)$ *and* $\delta_2 = -\frac{1}{6}$*. In particular,*

- *If* $x \in (0, \pi/2)$*, then the inequality*

$$\left(1 - \frac{2x^2}{3\pi^2}\right) + \frac{1}{3}\left(\frac{4}{\pi^2} - 1\right)(1-\cos x) < \frac{\sin x}{x} < \left(1 - \frac{2x^2}{3\pi^2}\right) + \left(\frac{2}{\pi} - \frac{5}{6}\right)(1-\cos x) \tag{3.3}$$

  holds with the optimal constants $\frac{1}{3}\left(\frac{4}{\pi^2} - 1\right)$ *and* $\left(\frac{2}{\pi} - \frac{5}{6}\right)$ *respectively.*

- *If* $x \in (0, \pi)$*, then the inequality*

$$\left(1 - \frac{2x^2}{3\pi^2}\right) + \frac{1}{3}\left(\frac{4}{\pi^2} - 1\right)(1-\cos x) < \frac{\sin x}{x} < \left(1 - \frac{2x^2}{3\pi^2}\right) - \frac{1}{6}(1-\cos x) \tag{3.4}$$

  holds with the optimal constants $\frac{1}{3}\left(\frac{4}{\pi^2} - 1\right)$ *and* $-\frac{1}{6}$ *respectively.*

Proof. Suppose

$$g(x) = \frac{\left[\frac{\sin x}{x} - \left(1 - \frac{2x^2}{3\pi^2}\right)\right]}{(1-\cos x)} = \frac{g_1(x)}{g_2(x)}; \quad x \in (0,\pi),$$

where $g_1(x) = \frac{\sin x}{x} - \left(1 - \frac{2x^2}{3\pi^2}\right)$ and $g_2(x) = 1 - \cos x$ such that $g_1(0+) = 0 = g_2(0)$. Differentiation yields

$$\frac{g_1'(x)}{g_2'(x)} = \frac{1}{x}\cot x - \frac{1}{x^2} + \frac{4}{3\pi^2}\frac{x}{\sin x}.$$

Based on series expansions in (2.1) and (2.2), we write

$$\frac{g_1'(x)}{g_2'(x)} = \frac{1}{x^2} - \sum_{k=1}^{\infty}\frac{2^{2k}}{(2k)!}|B_{2k}|x^{2k-2} - \frac{1}{x^2} + \frac{4}{3\pi^2} + \frac{4}{3\pi^2}\sum_{k=1}^{\infty}\frac{2(2^{2k-1}-1)}{(2k)!}|B_{2k}|x^{2k}.$$

After the rearrangement of terms, we equivalently get

$$\begin{aligned}\frac{g_1'(x)}{g_2'(x)} &= \frac{4}{3\pi^2} + \frac{4}{3\pi^2}\sum_{k=1}^{\infty}\frac{2(2^{2k-1}-1)}{(2k)!}|B_{2k}|x^{2k} - \sum_{k=0}^{\infty}\frac{2^{2k+2}}{(2k+2)!}|B_{2k+2}|x^{2k} \\ &= \left(\frac{4}{3\pi^2} - \frac{1}{3}\right) + \sum_{k=1}^{\infty}\left[\frac{8}{3\pi^2}\frac{(2^{2k-1}-1)}{(2k)!}|B_{2k}| - \frac{2^{2k+2}}{(2k+2)!}|B_{2k+2}|\right]x^{2k}.\end{aligned}$$

Then

$$\left(\frac{g_1'(x)}{g_2'(x)}\right)' = \sum_{k=1}^{\infty}\frac{16k}{(2k)!}\left[\frac{(2^{2k-1}-1)}{3\pi^2}|B_{2k}| - \frac{2^{2k-1}}{(2k+2)(2k+1)}|B_{2k+2}|\right]x^{2k-1}$$

To this end, for the positivity of the terms of the series, we need to prove that

$$\frac{(2^{2k-1}-1)}{3\pi^2}|B_{2k}| > \frac{2^{2k-1}}{(2k+2)(2k+1)}|B_{2k+2}|,$$

i.e.,

$$\frac{|B_{2k+2}|}{|B_{2k}|} < \frac{(2k+1)(2k+2)}{\pi^2}\frac{(2^{2k-1}-1)}{3\cdot 2^{2k-1}}; \quad k = 1,2,3,\cdots. \tag{3.5}$$

The inequality (3.5) is true for $k=1$ by virtue of absolute Bernoulli numbers $|B_2| = 1/6$ and $|B_4| = 1/30$. Now it remains to prove (3.5) for $k = 2,3,4,\cdots$. Because of the right inequality of Lemma 2.3, the relation (3.5) will be proved for $k = 2,3,4,\cdots$ if

$$\frac{2^{2k}-1}{2^{2k+2}-1} < \frac{2^{2k-1}-1}{3\cdot 2^{2k-1}}; \quad k = 2,3,4,\cdots,$$

i.e.,

$$3 \cdot 2^{4k-1} - 3 \cdot 2^{2k-1} < 2^{4k+1} - 2^{2k-1} - 2^{2k+2} + 1$$

or

$$2^{2k} + 2^{2k+3} < 2^{4k} + 3 \cdot 2^{2k} + 2$$

which is equivalent to $3 \cdot 2^{2k+1} < 2^{4k} + 2$. The last relation holds for $k = 2, 3, 4, \cdots$. Thus the derivative of $g_1'(x)/g_2'(x)$ is positive, it is strictly increasing in $(0, \pi)$. By Lemma 2.1, $g(x)$ is also strictly increasing in $(0, \pi)$. As a result, for $x \in (0, \pi/2)$, we have

$$\frac{1}{3}\left(\frac{4}{\pi^2} - 1\right) = \lim_{x \to 0^+} g(x) < g(x) < \lim_{x \to \pi/2^-} g(x) = \left(\frac{2}{\pi} - \frac{5}{6}\right).$$

This gives inequality (3.3). Similarly for $x \in (0, \pi)$, we have

$$\frac{1}{3}\left(\frac{4}{\pi^2} - 1\right) = \lim_{x \to 0^+} g(x) < g(x) < \lim_{x \to \pi^-} g(x) = -\frac{1}{6}.$$

This gives inequality (3.4) and the proof of Theorem 3.2 is completed. □

It is clear that the upper bound of $(\sin x)/x$ in (3.3) is sharper than the corresponding upper bound in (1.2). Let us now compare the bounds of $(\sin x)/x$ in (1.3) and (3.3) graphically. We consider the difference functions

$$l(x) = \left(1 - \frac{2x^2}{3\pi^2}\right) + \frac{1}{3}\left(\frac{4}{\pi^2} - 1\right)(1 - \cos x) - \left(1 - \frac{x^2}{6}\right),$$

and

$$u(x) = \left(1 - \frac{4x^2}{3\pi^2}\right) - \left(1 - \frac{2x^2}{3\pi^2}\right) - \left(\frac{2}{\pi} - \frac{5}{6}\right)(1 - \cos x).$$

The aforementioned difference functions are displayed in Figure 2.

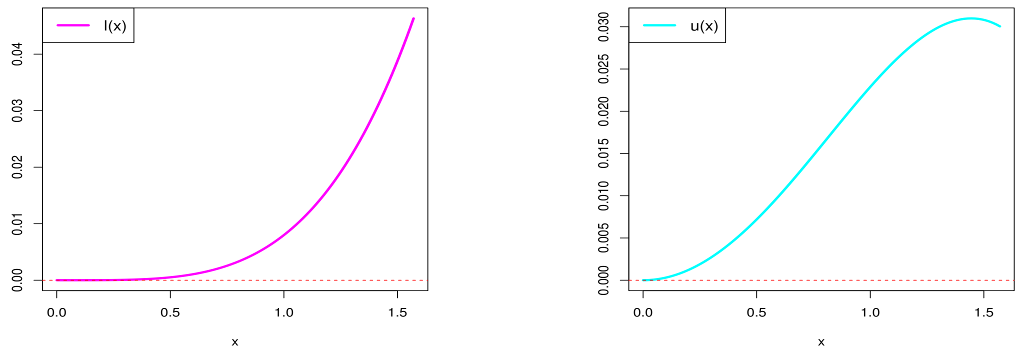

Figure 2. Plots of $l(x)$ and $u(x)$ for $x \in (0, \pi/2)$

From Figure 2, one can see that the bounds of $(\sin x)/x$ in (3.3) are superior to those in (1.3) in terms of sharpness.

Lastly, asking which bounds of $(\sin x)/x$ are better for $x \in (0, \pi)$ is natural. Numerical calculations and graphical comparisons reveal the following conclusions:

- There is no strict comparison between the corresponding bounds of $(\sin x)/x$ in (3.2) and (3.4).
- The lower bound of $(\sin x)/x$ in (3.4) is sharper than that in (3.2) except in the interval $(0, \gamma)$, where $\gamma \approx 0.257$.
- The upper bound of $(\sin x)/x$ in (3.4) is sharper than in (3.2) except in the interval $(0, \zeta)$, where $\zeta \approx 0.724$.

The above conclusions are supported by Figure 3 and Figure 4, where the curves of the difference functions

$$l'(x) = \left(1 - \frac{2x^2}{3\pi^2}\right) + \frac{1}{3}\left(\frac{4}{\pi^2} - 1\right)(1 - \cos x) - \left(1 - \frac{x^2}{6}\right) - \frac{1}{30}(1 - \cos x)^2$$

and

$$u'(x) = \left(1 - \frac{x^2}{6}\right) + \frac{1}{4}\left(\frac{\pi^2}{6} - 1\right)(1 - \cos x)^2 - \left(1 - \frac{2x^2}{3\pi^2}\right) + \frac{1}{6}(1 - \cos x)$$

are plotted.

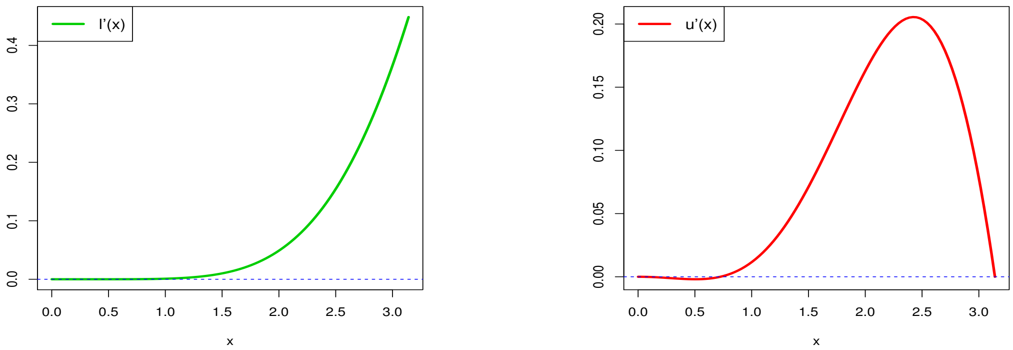

Figure 3. Plots of $l'(x)$ and $u'(x)$ for $x \in (0, \pi)$

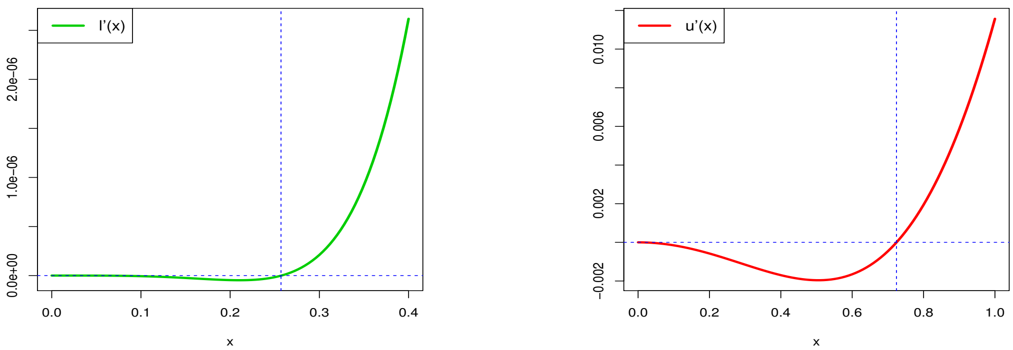

Figure 4. Plots of $l'(x)$ for $x \in (0, 0.4)$ and $u'(x)$ for $x \in (0, 1)$

4. An application to sinc integrals

There are several standard methods to evaluate the integral of sinc function over the set of non-negative real numbers. The value of this so-called Dirichlet integral $\int_0^\infty \frac{\sin x}{x} dx$ is $\pi/2$. However, it is not easy to evaluate $\int_0^r \frac{\sin x}{x} dx$, for any $r > 0$. In such a case, an alternative way is to approximate the concerned sinc integral. By using our main results, we can approximate $\int_0^r \frac{\sin x}{x} dx$, where $0 < r \leq \pi$. Because of sharpness, we can integrate the double inequality (3.4) over $[0, r]$. In particular, we have the following:

$$I_1 = \frac{(11\pi + 12)}{36} + \frac{(2\pi - 4)}{3\pi^2} < \int_0^{\pi/2} \frac{\sin x}{x} dx < \frac{(7\pi + 3)}{18} = I_2$$

and

$$I_3 = \frac{4\pi}{9} + \frac{4}{3\pi} < \int_0^{\pi} \frac{\sin x}{x} dx < \frac{33\pi}{54} = I_4.$$

Therefore, the sinc integrals can be approximated as:

$$\int_0^{\pi/2} \frac{\sin x}{x} dx \approx \frac{I_1 + I_2}{2} \approx 1.379387$$

and

$$\int_0^{\pi} \frac{\sin x}{x} dx \approx \frac{I_3 + I_4}{2} \approx 1.870269.$$

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