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ON SOME SHIFT INVARIANT MULTIVARIATE, INTEGRAL OPERATORS, REVISITED

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ABSTRACT. In recent articles the first author and H. Gonska (e.g., see [1], [2], [3]) studied global smoothness preservation by some univariate and multivariate linear operators over compact domains and $\mathbb{R}^d$, $d \geq 1$. Especially they studied a very general positive linear multivariate integral type operator (see [3]) over $\mathbb{R}^d$ that was introduced through a convolution like integration of another general positive multivariate linear operator with a scaling type function. In this paper the authors among others extend and generalize [3]. Also certain new similar, but more general, multivariate integral operators are introduced and studied. These operators come up naturally. And for all these are given sufficient conditions for multivariate: shift invariance, preservation of higher order global smoothness and sharpness of the related inequalities, convergence to the unit using the first modulus of continuity with respect to uniform norm, shape preserving on $\mathbb{R}^d$, and preservation of multivariate continuous probabilistic distribution functions. Several examples of very general specific multivariate integral operators satisfying this theory are given at the end.

1 INTRODUCTION

In approximating a function $f \in C(\mathbb{R}^d)$, $d \geq 1$, by $\mathcal{L}_k(f)$, where $\mathcal{L}_k$ are linear approximation operators, it is of importance to examine which features of $\mathcal{L}_k(f)$ are similar to the main characteristics of f. Also it is important to set sufficient conditions for suitable $\mathcal{L}_k$, so that $\mathcal{L}_k(f)$ best fit and resemble f as much as possible. I.e., how to choose the best approximators $\mathcal{L}_k(f)$ to f. In this paper we accomplish the above over $\mathbb{R}^d$, precisely we extend and generalize [3], also we are motivated by [3], [4], [5], and [6]. In particular, we introduce some new very general operators and study them thoroughly.

All the operators $\mathcal{L}_k$ studied here are of multivariate integral form, defined through convolution type integrations on $\mathbb{R}^d$, which involve other basic multivariate positive linear operators ℓ_k and a general scaling function φ. These operators are studied thoroughly in terms of multivariate: shift invariance, higher order global smoothness and shape preservation on $\mathbb{R}^d$, convergence to the unit operator with rates, sharpness of associated like to describe the basic properties of an example operator from [3] which is very simply introduced. Let φ be a real valued function of compact support $\subseteq \times_{i=1}^{d}[-a_i, a_i]$, $a_i > 0$. We assume that $\varphi \geq 0$, φ is Lebesgue measurable and

$$\int_{\mathbb{R}^d} \varphi(x-u)\, du = 1, \quad \forall\, x \in \mathbb{R}^d.$$

Let $f \in C(\mathbb{R}^d)$ that is uniformly continuous on $\mathbb{R}^d$. Define

$$(B_k f)(x) := \int_{\mathbb{R}^d} f\left(\frac{u}{2^k}\right) \varphi(2^k x - u)\, du, \quad k \in \mathbb{Z},\ x \in \mathbb{R}^d.$$

Then from [3], B_k is a shift invariant operator, $\omega_1^*(B_k f;\delta) \leq \omega_1^*(f;\delta)$, $\delta > 0$, where

$$\omega_1^*(f;\delta) := \sup_{\substack{x,y \in \mathbb{R}^d \\ \|x-y\| \leq \delta}} |f(x) - f(y)|,$$

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$\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^d$. The last inequality of global smoothness is sharp, namely is attained by projection functions. And we have convergence of B_k to the unit operator from

$$|B_k(f;x) - f(x)| \le \omega^*_{1,\infty}\left(f, \frac{\alpha}{2^k}\right),$$

where $\alpha := \max_{1\le i\le d}(a_i)$, $\omega^*_{1,\infty}$ is ω^*_1 with respect to $\|\cdot\|_\infty$.

Assume more now that φ is bounded on $\times_{i=1}^d[-a_i, a_i]$. Let $i_1, \dots, i_k \in \{1, \dots, d\}$ be such that $i_1 < \cdots < i_k$; $j_1, \dots, j_k \in \mathbb{Z}_+$. Assume that for $f \in C(\mathbb{R}^d)$: $\frac{\partial^{\sum_{\bar{k}=1}^k j_{\bar{k}}}}{\partial x_{i_1}^{j_1}\cdots\partial x_{i_k}^{j_k}} f(x)$ exists, is continuous and ≥ 0, for all $x \in \mathbb{R}^d$. Then

$$\frac{\partial^{\sum_{\bar{k}=1}^k j_{\bar{k}}}}{\partial x_1^{j_1}\cdots\partial x_{i_k}^{j_k}} B_k f(x) \ge 0, \quad \text{all } k \in \mathbb{Z},\ x \in \mathbb{R}^d.$$

Also for φ continuous on $\times_{i=1}^d[-a_i, a_i]$, B_k maps multivariate continuous probabilistic distribution functions to multivariate continuous probabilistic distribution functions.

2 Main Results

Let $X := C_U(\mathbb{R}^d)$, $d \ge 1$, be the space of uniformly continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$ and $C(\mathbb{R}^d)$ the space of continuous functions from $\mathbb{R}^d$ into $\mathbb{R}$. Here we need to introduce the following multivariate modulus of smoothness defined by

$$\omega_p(f;\delta) := \sup_{0\le h\le\delta} \|\Delta_h^p f(x)\|_\infty, \tag{2.1}$$

where

$$\Delta_h^p f(x) := \sum_{i=0}^p (-1)^{p-i}\binom{p}{i} f(x+ih), \tag{2.2}$$

$x := (x_1, \dots, x_d)$, $h := (h_1, \dots, h_d) \in \mathbb{R}^d$, $\delta := (\delta_1, \dots, \delta_d) \in \mathbb{R}^d$, $0 \le h \le \delta$ means $0 \le h_i \le \delta_i$, $i = \overline{1,p}$, $p \in \mathbb{N}$, $\|f\|_\infty := \sup\{|f(x_1, \dots, x_d)|;\, x_i \in \mathbb{R},\, i = \overline{1,d}\}$. For any $f \in X$ (see also [3]) one easily obtains that $\omega_p(f;\delta) < \infty$ for all $\delta \ge 0$, $\delta \in \mathbb{R}^d$. Let $\{\ell_k\}_{k\in\mathbb{Z}}$ be a sequence of positive linear operators that map X into $C(\mathbb{R}^d)$ with the property:

$$\ell_k(f;x) = \ell_0(f(2^{-k}\cdot);x), \quad \text{all } x \in \mathbb{R}^d,\ f \in X. \tag{2.3}$$

Next we assume that ($a > 0$ fixed, $a \in \mathbb{R}$)

$$\sup_{\substack{u,y\in\mathbb{R}^d\\ \|u-y\|_\infty\le a}} |\ell_0(f;u) - f(y)| \le \omega_{1,\infty}\left(f; \frac{ma+n}{2^r}\right) \tag{2.4}$$

is true for any $f \in X$, $m \in \mathbb{N}$, $n \in \mathbb{Z}_+$, $r \in \mathbb{Z}$, $\|u-y\|_\infty := \max\{|u_i - y_i|;\, i = \overline{1,d}\}$. Here

$$\omega_{1,\infty}(f,\gamma) := \sup_{\substack{x,y\in\mathbb{R}^d\\ \|x-y\|_\infty\le\gamma}} |f(x) - f(y)|, \tag{2.5}$$

where $\gamma \ge 0$, $\gamma \in \mathbb{R}$, is the first modulus of continuity of f with respect to supremum norm $\|\cdot\|_\infty$.

Let φ be a real valued function of compact support $\subseteq \times_{i=1}^d[-a_i, a_i]$, $a_i > 0$. We assume that $\varphi \ge 0$, φ is Lebesgue measurable and

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \varphi(x_1 - u_1, x_2 - u_2, \dots, x_d - u_d)\, du_1\cdots du_d = 1, \tag{2.6}$$

for all $(x_1, \dots, x_d) \in \mathbb{R}^d$. In brief

$$\int_{-\infty}^{+\infty} \varphi(x-u)\,du = 1, \tag{2.7}$$

for all $x \in \mathbb{R}^d$, where $u := (u_1, \dots, u_d)$, $du := du_1 \cdots du_d$,

$$\int_{-\infty}^{+\infty} := \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{d\text{-fold}}.$$

One can easily see that

$$\int_{-\infty}^{+\infty} \varphi(u)\,du = 1. \tag{2.8}$$

For examples of φ, see [3].

Let $\{\mathcal{L}_k\}_{k\in\mathbb{Z}}$ be the sequence of positive linear operators acting on X and defined by

$$\begin{aligned} \mathcal{L}_k(f; x_1, \dots, x_d) \quad := \quad & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} (\ell_k f)(u_1, \dots, u_d) \\ & \cdot \varphi(2^k x_1 - u_1, \dots, 2^k x_d - u_d) \cdot du_1 \cdots du_d, \\ & \text{all } (x_1, \dots, x_d) \in \mathbb{R}^d. \end{aligned} \tag{2.9}$$

In brief

$$\mathcal{L}_k(f; x) = \int_{-\infty}^{+\infty} \ell_k(f; u) \varphi(2^k x - u)\,du, \tag{2.10}$$

and

$$\mathcal{L}_0(f; x) = \int_{-\infty}^{+\infty} \ell_0(f; u) \varphi(x-u)\,du, \quad \text{all } x \in \mathbb{R}^d. \tag{2.11}$$

By (2.3) we have that

$$\mathcal{L}_k(f; x) = \mathcal{L}_0(f(2^{-k}\cdot); 2^k x), \quad \text{for all } x \in \mathbb{R}^d. \tag{2.12}$$

We need

Definition 1. *Let $f_\alpha(\cdot) := f(\cdot + \alpha)$, $\alpha \in \mathbb{R}^d$, and ϕ be an operator. If $\phi(f_\alpha) = (\phi f)_\alpha$, then ϕ is called a shift invariant operator.*

First we generalize Theorem 1 of [3] on *global smoothness preservation by $\mathcal{L}_k$*.

Theorem 2.1. *For any $f \in X$, assume that*

$$|(\Delta_h^p \ell_0(f))(x-u)| \le \omega_p(f; h), \quad \forall x, u \in \mathbb{R}^d, \tag{2.13}$$

$h > 0$, $h \in \mathbb{R}^d$, $p \in \mathbb{N}$. Then

$$\omega_p(\mathcal{L}_k f; \delta) \le \omega_p(f; \delta), \quad \forall \delta > 0,\ \delta \in \mathbb{R}^d,\ k \in \mathbb{Z}. \tag{2.14}$$

Proof. We have

$$\begin{aligned}
|(\Delta_h^p(\mathcal{L}_0(f)))(x)| &= \left|\sum_{j=0}^{p}(-1)^{p-j}\binom{p}{j}\mathcal{L}_0(f;x+jh)\right| \\
&= \left|\sum_{j=0}^{p}(-1)^{p-j}\binom{p}{j}\int_{-\infty}^{+\infty}\ell_0(f;x+jh-u)\varphi(u)\,du\right| \\
&= \left|\int_{-\infty}^{+\infty}((\Delta_h^p(\ell_0(f)))(x-u))\varphi(u)\,du\right| \\
&\le \int_{-\infty}^{+\infty}|(\Delta_h^p(\ell_0(f)))(x-u)|\varphi(u)\,du \underset{(2.8)}{\overset{(2.13)}{\le}} \omega_p(f;h),
\end{aligned}$$

all $0 < h \le \delta$; $h, \delta, x \in \mathbb{R}^d$. By (2.12) and (2.2) we get

$$|(\Delta_h^p\mathcal{L}_k(f))(x)| = |(\Delta_{2^k h}^p\mathcal{L}_0(f(2^{-k}\cdot)))(2^k x)| \le \omega_p(f(2^{-k}\cdot);2^k h) = \omega_p(f;h).$$

∎

We need Theorem 2 from [3]:

Theorem 2.2. *For $f \in X$, under the assumption (2.4), it holds*

$$|\mathcal{L}_k(f;x) - f(x)| \le \omega_{1,\infty}\left(f;\frac{ma+n}{2^{k+r}}\right), \tag{2.15}$$

where $m \in \mathbb{N}$, $n \in \mathbb{Z}_+$, $k, r \in \mathbb{Z}$, and $\alpha := \max(a_i)$, $i = 1, \dots, n$.

Next we define the following new operators

$$\mathcal{L}_{k,j}(f;x) := \int_{-\infty}^{+\infty}\ell_k(f;2^k x - ju)\varphi(u)\,du, \quad k \in \mathbb{Z},\ j \in \mathbb{N},\ x \in \mathbb{R}^d. \tag{2.16}$$

Notice that

$$\mathcal{L}_{k,j}(f;x) = \mathcal{L}_{0,j}(f(2^{-k}\cdot);2^k x), \tag{2.17}$$

all $x \in \mathbb{R}^d$. And

$$\mathcal{L}_{k,1} = \mathcal{L}_k. \tag{2.18}$$

One can easily see that

$$\mathcal{L}_{k,j}(f;x) = \int_{-\infty}^{+\infty}(\ell_k f)(u)\frac{1}{j^d}\varphi\left(\frac{1}{j}(2^k x - u)\right)du, \quad k \in \mathbb{Z},\ x \in \mathbb{R}^d. \tag{2.19}$$

By (2.7) we see that

$$\int_{-\infty}^{+\infty}\frac{1}{j^d}\varphi\left(\frac{1}{j}(x-u)\right)du = 1, \quad \text{all } j \in \mathbb{N},\ x \in \mathbb{R}^d,\ d \ge 1. \tag{2.20}$$

Call

$$\varphi_j^*(\cdot) := \frac{1}{j^d}\varphi\left(\frac{1}{j}\cdot\right), \quad j \in \mathbb{N}, \tag{2.21}$$

then supp $\varphi_j^* \subseteq \times_{i=1}^d[-ja_i, ja_i]$, $a_i > 0$. Furthermore φ_j^* has also all other properties of φ.

Clearly now one can see that the operator $\mathcal{L}_{k,j}$ based on φ is identical to the operator $\mathcal{L}_k$ based on φ_j^*. According to this last comment we present

Theorem 2.3. *Assume that*

$$\ell_0(f(2^{-k}\cdot+\alpha);2^k u)=\ell_0(f(2^{-k}\cdot);2^k(u+\alpha)), \tag{2.22}$$

for all $k\in\mathbb{Z}$, $\alpha\in\mathbb{R}^d$ *fixed, all* $u\in\mathbb{R}^d$*; any* $f\in X$*. Then* $\mathcal{L}_{k,j}$ *is a shift invariant operator, all* $k\in\mathbb{Z}$, $j\in\mathbb{N}$.

Proof. See [3], Proposition 1. ∎

Next we study the global smoothness preservation property of $\mathcal{L}_{k,j}$.

Theorem 2.4. *Assume* (13) *is true, then*

$$\omega_p(\mathcal{L}_{k,j}f;\delta)\le\omega_p(f;\delta),\quad \forall\delta>0,\ \delta\in\mathbb{R}^d,\ k\in\mathbb{Z},\ j\in\mathbb{N}. \tag{2.23}$$

Proof. See Theorem 2.1. ∎

Convergence with rates of $\mathcal{L}_{k,j}$ operators follows.

Theorem 2.5. *For* $f\in X$*, under the assumption* (2.4)*, it holds*

$$|\mathcal{L}_{k,j}(f;x)-f(x)|\le\omega_{1,\infty}\left(f;\frac{mja+n}{2^{k+r}}\right), \tag{2.24}$$

where $m,j\in\mathbb{N}$, $n\in\mathbb{Z}_+$, $k,r\in\mathbb{Z}$*, and* $a:=\max(a_i)$, $i=1,\dots,n$.

Proof. See Theorem 2 of [3]. ∎

In what follows we generalize the shift-invariant multiple integral operators presented earlier by using the idea which produces the generalized Jackson's operators in the classical Approximation Theory.

Define

$$I_{0,q}(f;x):=-\int_{-\infty}^{+\infty}\sum_{j=1}^{q}(-1)^j\binom{q}{j}\ell_0(f;x-ju)\varphi(u)\,du, \tag{2.25}$$

and

$$I_{k,q}(f;x):=-\int_{-\infty}^{+\infty}\sum_{j=1}^{q}(-1)^j\binom{q}{j}\ell_k(f;2^kx-ju)\varphi(u)\,du,\quad k\in\mathbb{Z},\ q\in\mathbb{N},\ \text{all } x\in\mathbb{R}^d. \tag{2.26}$$

Notice by (2.3) that

$$I_{k,q}(f;x)=I_{0,q}(f(2^{-k}\cdot);2^kx),\quad \text{all } x\in\mathbb{R}^d. \tag{2.27}$$

Precisely we see that

$$I_{0,q}(f;x)=-\sum_{j=1}^{q}(-1)^j\binom{q}{j}\mathcal{L}_{0,j}(f;x), \tag{2.28}$$

and

$$I_{k,q}(f;x)=-\sum_{j=1}^{q}(-1)^j\binom{q}{j}\mathcal{L}_{k,j}(f;x),\quad \text{all } x\in\mathbb{R}^d. \tag{2.29}$$

We give

Theorem 2.6. *Assume that* (2.22) *is true. Then* $I_{k,q}$ *is a shift-invariant operator.*

Proof. Directly from Theorem 2.3. ■

Global smoothness of $I_{k,q}$ operators follows.

Theorem 2.7. *Assume that* (2.13) *is true, then*

$$\omega_p(I_{k,q}(f);\delta) \le (2^q-1)\omega_p(f;\delta), \quad \delta>0,\ p,q\in\mathbb{N},\ k\in\mathbb{Z}. \tag{2.30}$$

Proof. Application of Theorem 2.4, and see that $\sum_{j=1}^{q}\binom{q}{j} = 2^q-1$. ■

Also notice that

$$-\sum_{j=1}^{q}(-1)^j\binom{q}{j} = 1. \tag{2.31}$$

Next we study the convergence to the unit of $I_{k,q}$ operators.

Theorem 2.8. *Let* $f\in X$ *and assume that* (4) *holds. Then*

$$|I_{k,q}(f,x)-f(x)| \le (2^q-1)\omega_{1,\infty}\left(f;\frac{mqa+n}{2^{k+r}}\right), \tag{2.32}$$

where $k,r\in\mathbb{Z}$, $q,m\in\mathbb{N}$, $n\in\mathbb{Z}_+$, *and* $a:=\max(a_i)$, $i=1,\dots,n$.

Proof. We see that

$$\begin{aligned}
|I_{k,q}(f,x)-f(x)| &\overset{(2.31)}{=} \left|-\sum_{j=1}^{q}(-1)^j\binom{q}{j}\mathcal{L}_{k,j}(f;x) - \left(-\sum_{j=1}^{q}(-1)^j\binom{q}{j}\right)f(x)\right| \\
&= \left|\sum_{j=1}^{q}(-1)^j\binom{q}{j}\left(\mathcal{L}_{k,j}(f;x)-f(x)\right)\right| \\
&\le \sum_{j=1}^{q}\binom{q}{j}\left|\mathcal{L}_{k,j}(f;x)-f(x)\right| \\
&\overset{(2.24)}{\le} \sum_{j=1}^{q}\binom{q}{j}\omega_{1,\infty}\left(f;\frac{mja+n}{2^{k+r}}\right) \\
&\le (2^q-1)\omega_{1,\infty}\left(f;\frac{mqa+n}{2^{k+r}}\right). \qquad \blacksquare
\end{aligned}$$

About the sharpness of the above global smoothness inequalities (2.23), (2.30) follow:

Proposition 2.1. *Let* $d\ge 2$. *For* $i\in\{1,\dots,d\}$, *let* $pr_i:\mathbb{R}^d\ni(x_1,\dots,x_d)\to x_i\in\mathbb{R}$, *denote the projection onto the* i*th coordinate. Assume that for at least one* i *one has*

$$\ell_0(pr_i(2^{-k}\cdot);2^kx-ju)-\ell_0(pr_i(2^{-k}\cdot);2^ky-ju)=x_i-y_i, \tag{2.33}$$

for all $x,y,u\in\mathbb{R}^d$, *for a fixed* $j\in\mathbb{N}$. *Then*

$$\omega_1(\mathcal{L}_{k,j}pr_i;\delta)=\omega_1(pr_i;\delta), \tag{2.34}$$

any $\delta>0$, $\delta\in\mathbb{R}^d$, *proving inequality* (2.23) *to be sharp.*

Proof. Note that pr_i is a uniformly continuous function on $\mathbb{R}^d$. Furthermore,

$$\begin{aligned}
\mathcal{L}_{k,j}(pr_i;x+h)-\mathcal{L}_{k,j}(pr_i;x) &= \int_{-\infty}^{+\infty}[\ell_0(pr_i(2^{-k}\cdot);2^k(x+h)-ju) \\
&\quad -\ell_0(pr_i(2^{-k}\cdot);2^kx-ju)]\varphi(u)\,du \\
&\overset{(2.33)}{=} \int_{-\infty}^{+\infty}h_i\varphi(u)\,du \overset{(2.8)}{=} h_i.
\end{aligned}$$

That is,

$$(\Delta_h^1 \mathcal{L}_{k,j}(pr_i))(x) = (\Delta_h^1 pr_i)(x), \quad \text{all } x \in \mathbb{R}^d. \tag{2.35}$$

Then

$$\|(\Delta_h^1 \mathcal{L}_{k,j}(pr_i))(x)\|_\infty = \|(\Delta_h^1 pr_i)(x)\|_\infty$$

and

$$\omega_1(\mathcal{L}_{k,j} pr_i; \delta) = \omega_1(pr_i; \delta),$$

any $\delta > 0$, $\delta \in \mathbb{R}^d$. ∎

Proposition 2.2. *Under the assumptions and notations of Proposition* 2.1*,* $j = 1, \dots, q$ *we get that*

$$\omega_1(I_{k,q} pr_i; \delta) = \omega_1(pr_i; \delta), \tag{2.36}$$

any $\delta > 0$*,* $\delta \in \mathbb{R}^d$*. Establishing that in some cases we can do better than inequality* (2.30)*.*

Proof. We have

$$\begin{aligned} (\Delta_h^1 I_{k,q}(pr_i))(x) &= -\sum_{j=1}^{q} (-1)^j \binom{q}{j} (\Delta_h^1 \mathcal{L}_{k,j}(pr_i))(x) \\ &\overset{(2.35)}{=} \left(-\sum_{j=1}^{q} (-1)^j \binom{q}{j} \right) \cdot (\Delta_h^1 pr_i)(x) \\ &\overset{(2.31)}{=} (\Delta_h^1 pr_i)(x). \end{aligned}$$

That is

$$\|(\Delta_h^1 I_{k,q}(pr_i))(x)\|_\infty = \|(\Delta_h^1 pr_i)(x)\|_\infty,$$

establishing (2.36). ∎

Remark 2.1. Under the assumptions and notations of Propositions 2.1 and 2.2 when $p > 1$ inequalities (2.23) and (2.30) are trivially attained by pr_i, respectively.

Next we study the differentiability of $\mathcal{L}_{k,j}$, $I_{k,q}$ operators.

Theorem 2.9. *Let* $\rho \in \mathbb{N}$ *and assume that* ℓ_0 *maps* $C^{(\rho)}(\mathbb{R}^d)$ *into itself. Additionally assume that* φ *is bounded. Let* $i_1, \dots, i_k \in \{1, \dots, d\}$ *be such that* $i_1 < \cdots < i_k$*, and* $j_1, \dots, j_k \in \mathbb{Z}_+$*. Suppose that* $(\ell_0 f)^\partial(x) := \frac{\partial^{\sum_{\tilde{k}=1}^k j_{\tilde{k}}}}{\partial x_1^{j_1} \cdots \partial x_{i_k}^{j_k}} \ell_0(f; x)$ *exists and is continuous for all* $x \in \mathbb{R}^d$*. Then*

$$(\mathcal{L}_{k,j} f)^\partial(x) := \frac{\partial^{\sum_{\tilde{k}=1}^k j_{\tilde{k}}}}{\partial x_1^{j_1} \cdots \partial x_{i_k}^{j_k}} \mathcal{L}_{k,j}(f; x) \tag{2.37}$$

exists, as well as $(I_{k,q} f)^\partial(x)$*, for all* $x \in \mathbb{R}^d$*;* $k \in \mathbb{Z}$*,* $j, q \in \mathbb{N}$*. More precisely we get that*

$$(\mathcal{L}_{k,j} f)^\partial(x) = 2^{k\rho} \int_{-\infty}^{+\infty} (\ell_k f)^\partial(2^k x - ju) \varphi(u)\, du, \tag{2.38}$$

and

$$(I_{k,q} f)^\partial(x) = -\sum_{j=1}^{q} (-1)^j \binom{q}{j} (\mathcal{L}_{k,j} f)^\partial(x), \tag{2.39}$$

all $x \in \mathbb{R}^d$*.*

Proof. By Theorem 3 of [3], definition (2.16) of $\mathcal{L}_{k,j}$, (2.21) and (2.29). ■

Remark 2.2. On Theorem 2.9. If $(\ell_{0,j}f)^{\partial}(x) \geq 0$ for all $x \in \mathbb{R}^d$, then $(\mathcal{L}_{0,j}f)^{\partial}(x) \geq 0$ and thus $(\mathcal{L}_{k,j}f)^{\partial}(x) \geq 0$, for all $x \in \mathbb{R}^d$.

Next we study the preservation of continuous probabilistic distribution functions on $\mathbb{R}^d$ by $\mathcal{L}_{k,j}$ operators.

Theorem 2.10. *Let ℓ_k be a positive linear operator from $C(\mathbb{R}^d)$ into itself as in (2.3), i.e., $\ell_k(f,x) = \ell_0(f(2^{-k}\cdot),x)$, for all $x \in \mathbb{R}^d$, here for all $f \in C(\mathbb{R}^d)$, $k \in \mathbb{Z}$. Assume that f is a probabilistic distribution function from $\mathbb{R}^d$ into $\mathbb{R}$ that is continuous. Assume that $\ell_0(f)$ is also a continuous probabilistic distribution function, whenever f is a continuous probabilistic distribution function. Assume furthermore, that $\varphi \geq 0$ is continuous on $\times_{i=1}^d[-a_i,a_i]$, $a_i > 0$, $\operatorname{supp}\varphi \subseteq \times_{i=1}^d[-a_i,a_i]$, $\int_{\mathbb{R}^d}\varphi(x-u)\,du = 1$, for all $x \in \mathbb{R}^d$.*

Then the operator

$$\mathcal{L}_{k,j}(f;x) = \int_{-\infty}^{\infty}(\ell_k f)(2^k x - ju)\varphi(u)\,du, \tag{2.40}$$

$k \in \mathbb{Z}$, $j \in \mathbb{N}$, when applied on f as above produces a continuous probabilistic distribution function from $\mathbb{R}^d$ into $\mathbb{R}$.

Proof. By Theorem 4 of [3], noticing again that the operator $\mathcal{L}_{k,j}$ based on φ is the operator $\mathcal{L}_k$ based on φ_j^* as in (2.21). ■

Remark 2.3. Observe that $\omega_p(F;\delta) < +\infty$ for any probability distribution function $F : \mathbb{R}^d \to [0,1]$. And of course all estimates involving ω_p in this paper apply also for just continuous probabilistic distribution functions on $\mathbb{R}^d$.

3 Applications

Next we discuss applications on Section 2. Here ℓ_k operators are specified. The basic function φ is as in Section 2. Only for the following operators $(A_{k,j})_{k\in\mathbb{Z}}$, φ will be assumed additionally to be an even continuous function $\varphi(-x) = \varphi(x)$, $\forall x \in \mathbb{R}^d$. Again define

$$\varphi_j^*(\cdot) := \frac{1}{j^d}\varphi\left(\frac{1}{j}\cdot\right), \quad j \in \mathbb{N}. \tag{3.1}$$

So we introduce the following multivariate operators, all defined for each $k \in \mathbb{Z}$, acting on X; $x \in \mathbb{R}^d$.

(i)

$$(A_{k,j}f)(x) := \int_{-\infty}^{+\infty} r_k^f(u)\varphi_j^*(2^k x - u)\,du, \tag{3.2}$$

where

$$r_k^f(u) := 2^{kd}\int_{-\infty}^{+\infty} f(t)\varphi_j^*(2^k t - u)\,dt, \quad u,t \in \mathbb{R}^d. \tag{3.3}$$

(ii)

$$(B_{k,j}f)(x) := \int_{-\infty}^{+\infty} f\left(\frac{u}{2^k}\right)\varphi_j^*(2^k x - u)\,du. \tag{3.4}$$

(iii)

$$(L_{k,j}f)(x) := \int_{-\infty}^{+\infty} c_k^f(u)\varphi_j^*(2^k x - u)\,du, \tag{3.5}$$

where

$$c_k^f(u) := 2^{kd} \int \cdots \int_{2^{-k}u_i}^{2^{-k}(u_i+1)} \cdots \int f(t)\,dt, \quad \text{all } u \in \mathbb{R}^d. \tag{3.6}$$

(iv)

$$(\Gamma_{k,j} f)(x) := \int_{-\infty}^{+\infty} \gamma_k^f(u) \varphi_j^*(2^k x - u)\,du, \tag{3.7}$$

where

$$\gamma_k^f(u) := \sum_{j_1=0}^{n_1} \cdots \sum_{j_d=0}^{n_d} w_{j_1,\dots,j_d} \cdot f\left(\frac{u_1}{2^k} + \frac{j_1}{2^k \cdot n_1}, \dots, \frac{u_d}{2^k} + \frac{j_d}{2^k n_d}\right), \tag{3.8}$$

$(n_1, \dots, n_d) \in \mathbb{N}^d$, $w_{j_1,\dots,j_d} \geq 0$, $\sum_{j_1=0}^{n_1} \cdots \sum_{j_1=0}^{n_d} w_{j_1,\dots,j_d} = 1$, all $u \in \mathbb{R}^d$.

Define also, for $k \in \mathbb{Z}$, $q \in \mathbb{N}$, all $x \in \mathbb{R}^d$, the corresponding related multivariate operators:

$$I_{k,q}^A(f;x) := -\sum_{j=1}^{q} (-1)^j \binom{q}{j} A_{k,j}(f;x), \tag{3.9}$$

$$I_{k,q}^B(f;x) := -\sum_{j=1}^{q} (-1)^j \binom{q}{j} B_{k,j}(f;x), \tag{3.10}$$

$$I_{k,q}^L(f;x) := -\sum_{j=1}^{q} (-1)^j \binom{q}{j} L_{k,j}(f;x), \tag{3.11}$$

and

$$I_{k,q}^\Gamma(f;x) := -\sum_{j=1}^{q} (-1)^j \binom{q}{j} \Gamma_{k,j}(f;x). \tag{3.12}$$

All the above operators fulfill the assumptions of the corresponding related theorems of Section 2. Therefore the results of Section 2 can be applied on all these example operators, as described above, and produce nice, specific and simpler, analogous results to [3] and [6].

Comment. All the results of this article, except about sharpness of global smoothness inequalities, are true also when $X = C_B(\mathbb{R}^d)$, the space of bounded continuous real valued functions on $\mathbb{R}^d$, $d \geq 1$.

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