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A PROOF OF THE ARITHMETIC MEAN-GEOMETRIC MEAN-HARMONIC MEAN INEQUALITIES

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Abstract. In the note, using Cauchy-Schwartz-Buniakowski's inequality, the authors give a new proof of the arithmetic mean-geometric mean-harmonic mean inequalities.

1 Introduction

The simplest and most classical mean values are the arithmetic, the geometric, and the harmonic mean values. For a positive sequence $a = (a_1, a_2, \dots, a_n)$, these mean values are defined respectively by

$$A_n(a) = \frac{1}{n}\sum_{i=1}^{n} a_i, \qquad G_n(a) = \sqrt[n]{\prod_{i=1}^{n} a_i}, \qquad H_n(a) = \frac{n}{\sum\limits_{i=1}^{n}\frac{1}{a_i}}. \tag{1.1}$$

For a positive integrable function f defined on $[x, y]$, their integral analogues of (1.1) are given by

$$A(f) = \frac{1}{y-x}\int_x^y f(t)\mathrm{d}t, \qquad G(f) = \exp\left(\frac{1}{y-x}\int_x^y \ln f(t)\mathrm{d}t\right), \qquad H(f) = \frac{y-x}{\displaystyle\int_x^y \frac{\mathrm{d}t}{f(t)}}. \tag{1.2}$$

It is well-known that

$$A_n(a) \geqslant G_n(a) \geqslant H_n(a), \quad A(f) \geqslant G(f) \geqslant H(f) \tag{1.3}$$

are called the arithmetic mean-geometric mean-harmonic mean inequalities.

For the sake of brevity, the inequality between the arithmetic and geometric means will be called A-G inequality, while the inequality between the geometric and harmonic means will be called G-H inequality.

The A-G inequality has found much interest among many mathematicians, and there are numerous new proofs, extensions, refinements, and variants of it. The study of the A-G inequality has a rich literature, for details, please refer to [2, 3, 4], and the like. Recently, H. Alzer [1] and J. Pečarić and S. Varošanec [6] gave two new proofs of the A-G inequality.

The concepts of mean values have been generalized, extended in many directons. A recent developments concerning the mean values has simply been introduced in [5, 7, 8, 9].

In this note, using Cauchy-Schwartz-Buniakowski's inequality, we give a new proof of the A-G-H inequalities.

2 A New Proof of the A-G-H Inequalities

For a continuous function f, define

$$\begin{aligned} \psi(r) &= \left(\frac{1}{y-x}\int_x^y f^r(t)\,\mathrm{d}t\right)^{1/r}, \quad r \neq 0; \\ \psi(0) &= G(f). \end{aligned} \tag{2.1}$$

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For a positive sequence $a = (a_1, a_2, \dots, a_n)$, define

$$\varphi(r) = \left(\frac{1}{n}\sum_{i=1}^{n} a_i^r\right)^{1/r}, \quad r \neq 0; \qquad \varphi(0) = G_n(a). \tag{2.2}$$

Theorem. *The functions $\psi(r)$ and $\varphi(r)$ are increasing with $r \in \mathbb{R}$, respectively.*

Proof. Simple calculation yields

$$\begin{aligned}\ln\psi(r) &= \frac{\ln\int_x^y f^r(t)\,\mathrm{d}t - \ln(y-x)}{r}\\ &= \frac{\ln\int_x^y f^r(t)\,\mathrm{d}t - \ln\int_x^y f^0(t)\,\mathrm{d}t}{r}\\ &= \frac{1}{r}\int_0^r \frac{\int_x^y f^s(t)\ln f(t)\,\mathrm{d}t}{\int_x^y f^s(t)\,\mathrm{d}t}\,\mathrm{d}s.\end{aligned}$$

The lemma 1 in [10] states that, if f is a differentiable and increasing function on a given interval I, then the arithmetic mean $\psi(r,s)$ of f defined as

$$\psi(r,s) = \frac{1}{s-r}\int_r^s f(t)dt, \quad r-s\neq 0, \qquad \psi(r,r) = f(r) \tag{2.3}$$

is also increasing with both r and s on I.

Therefore, it is sufficient to verify that

$$\mathcal{F}(s) \triangleq \frac{\int_x^y f^s(t)\ln f(t)\,\mathrm{d}t}{\int_x^y f^s(t)\,\mathrm{d}t}$$

is increasing in $s \in \mathbb{R}$.

Let $g(s) = \int_x^y f^s(t)\,\mathrm{d}t$, $s\in\mathbb{R}$. Then $\mathcal{F}(s)$ increases with s if and only if $g''(s)g(s) - \big[g'(s)\big]^2 \geqslant 0$, that is,

$$\left(\int_x^y f^s(t)\ln f(t)\,\mathrm{d}t\right)^2 \leqslant \int_x^y f^s(t)\,\mathrm{d}t \int_x^y f^s(t)\big[\ln f(t)\big]^2\,\mathrm{d}t. \tag{2.4}$$

Since

$$\int_x^y f^s(t)\ln f(t)\,\mathrm{d}t = \int_x^y f^{s/2}(t)\big[f^{s/2}(s)\ln f(t)\big]\,\mathrm{d}t,$$

from Cauchy-Schwartz-Buniakowski's integral inequality in integral form, the inequality (2.4) follows. The function $\psi(r)$ is increasing with r.

From straightforward computation, we have

$$\begin{aligned}\ln\varphi(r) &= \frac{1}{r}\left(\ln\sum_{i=1}^n a_i^r - \ln n\right)\\ &= \frac{1}{r}\left(\ln\sum_{i=1}^n a_i^r - \ln\sum_{i=1}^n a_i^0\right)\\ &= \frac{1}{r}\int_0^r\left(\sum_{i=1}^n a_i^s\ln a_i \Big/ \sum_{i=1}^n a_i^s\right)\mathrm{d}s.\end{aligned} \tag{2.5}$$

Using Cauchy-Schwartz-Buniakowski's inequality in discrete form, by the similar arguments as proving the monotonicity of $\psi(r)$, we can easily obtain that the function $\varphi(r)$ increases with r. The proof of Theorem follows. ∎

Corollary. *For a positive continuous function f or a positive sequence $a = (a_1, a_2, \dots, a_n)$, we have the following A-G-H inequalities:*

$$A(f) \geqslant G(f) \geqslant H(f), \quad A_n(a) \geqslant G_n(a) \geqslant H_n(a). \tag{2.6}$$

Proof. It is easy to see that $\psi(1) = A(f)$, $\psi(-1) = H(f)$, $\varphi(1) = A_n(a)$ and $\varphi(-1) = H_n(a)$. Thus, the A-G-H inequalities in integral form follows from the monotonicity of $\psi(r)$, the A-G-H inequalities in discrete form follows from the monotonicity of $\varphi(r)$. The proof is complete. ∎

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