

ON THE OSTROWSKI INTEGRAL INEQUALITY FOR LIPSCHITZIAN MAPPINGS AND APPLICATIONS

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ABSTRACT. A generalization of Ostrowski's inequality for lipschitzian mappings and applications in Numerical Analysis and for Euler's Beta function are given.

1 INTRODUCTION

The following theorem contains the integral inequality which is known in the literature as Ostrowski's inequality [2, p. 469].

Theorem 1.1. *Let $f : [a, b] \rightarrow \mathbf{R}$ be a differentiable mapping on (a, b) whose derivative is bounded on (a, b) and denote $\|f'\|_\infty = \sup_{t \in (a, b)} |f'(t)| < \infty$. Then for all $x \in [a, b]$ we have the inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty.$$

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

In this paper we prove that Ostrowski's inequality also holds for lipschitzian mappings and apply it in obtaining a Riemann's type quadrature formula for this class of mappings. Applications for Euler's Beta function are also given.

2 OSTROWSKI'S INEQUALITY FOR LIPSCHITZIAN MAPPINGS

The following inequality for lipschitzian mappings holds:

Theorem 2.1. *Let $u : [a, b] \rightarrow \mathbf{R}$ be an L -lipschitzian mapping on $[a, b]$, i.e.,*

$$|u(x) - u(y)| \leq L|x - y| \quad \text{for all } x, y \in [a, b].$$

Then we have the inequality

$$(2.1) \quad \left| \int_a^b u(t) dt - u(x)(b-a) \right| \leq L(b-a)^2 \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right].$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible one.

Proof. Using the integration by parts formula for Riemann-Stieltjes integral we have

$$\int_a^x (t-a) du(t) = u(x)(x-a) - \int_a^x u(t) dt$$

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and

$$\int_x^b (t-b) du(t) = u(x)(b-x) - \int_x^b u(t) dt.$$

If we add the above two equalities, we get

$$(2.2) \quad u(x)(b-a) - \int_a^b u(t) dt = \int_a^b p(x,t) du(t)$$

where

$$p(x,t) := \begin{cases} t-a & \text{if } t \in [a,x) \\ t-b & \text{if } x \in [x,b] \end{cases},$$

for all $x, t \in [a, b]$.

Now, assume that $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < \dots < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\nu(\Delta_n) \rightarrow 0$ as $n \rightarrow \infty$, where $\nu(\Delta_n) := \max_{i \in \{0, \dots, n-1\}} (x_{i+1}^{(n)} - x_i^{(n)})$ and $\xi_i^{(n)} \in [x_i^{(n)}, x_{i+1}^{(n)}]$. If $p : [a, b] \rightarrow \mathbf{R}$ is Riemann integrable on $[a, b]$ and $v : [a, b] \rightarrow \mathbf{R}$ is L -lipschitzian on $[a, b]$, then

$$(2.3) \quad \begin{aligned} \left| \int_a^b p(x) dv(x) \right| &= \left| \lim_{\nu(\Delta_n) \rightarrow 0} \sum_{i=0}^{n-1} p(\xi_i^{(n)}) [v(x_{i+1}^{(n)}) - v(x_i^{(n)})] \right| \\ &\leq \lim_{\nu(\Delta_n) \rightarrow 0} \sum_{i=0}^{n-1} |p(\xi_i^{(n)})| (x_{i+1}^{(n)} - x_i^{(n)}) \left| \frac{v(x_{i+1}^{(n)}) - v(x_i^{(n)})}{x_{i+1}^{(n)} - x_i^{(n)}} \right| \\ &\leq L \lim_{\nu(\Delta_n) \rightarrow 0} \sum_{i=0}^{n-1} |p(\xi_i^{(n)})| (x_{i+1}^{(n)} - x_i^{(n)}) = L \int_a^b |p(x)| dx. \end{aligned}$$

Applying the inequality (2.3) for $p(x, t)$ as above and $v(x) = u(x)$, $x \in [a, b]$, we get

$$(2.4) \quad \begin{aligned} \left| \int_a^b p(x,t) du(t) \right| &\leq L \int_a^b |p(x,t)| dt \\ &= L \left[\int_a^x |t-a| dt + \int_x^b |t-b| dt \right] = \frac{L}{2} [(x-a)^2 + (b-x)^2] \\ &= L(b-a)^2 \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] \end{aligned}$$

and then by (2.4), via the identity (2.2), we deduce the desired inequality (2.1).

Now, assume that the inequality (2.1) holds with a constant $C > 0$, i.e.,

$$(2.5) \quad \left| \int_a^b u(t) dt - u(x)(b-a) \right| \leq L(b-a)^2 \left[C + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right]$$

for all $x \in [a, b]$.

Consider the mapping $f : [a, b] \rightarrow \mathbf{R}$, $f(x) = x$ in (2.5). Then

$$\left| x - \frac{a+b}{2} \right| \leq C + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}$$

for all $x \in [a, b]$; and then for $x = a$, we get

$$\frac{b-a}{2} \leq \left(C + \frac{1}{4} \right) (b-a)$$

which implies that $C \geq \frac{1}{4}$ and the theorem is completely proved. ■

The following corollary holds:

Corollary 2.2. *Let $u : [a, b] \rightarrow \mathbf{R}$ be as above. Then we have the inequality:*

$$(2.6) \quad \left| \int_a^b u(t) dx - u\left(\frac{a+b}{2}\right)(b-a) \right| \leq \frac{1}{4}L(b-a)^2.$$

Remark 2.1. *It is well known that if $f : [a, b] \rightarrow \mathbf{R}$ is a convex mapping on $[a, b]$, then Hermite-Hadamard's inequality holds*

$$(2.7) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}.$$

Now, if we assume that $f : I \subset \mathbf{R} \rightarrow \mathbf{R}$ is convex on I and $a, b \in \text{Int}(I)$, $a < b$; then f'_+ is monotonous nondecreasing on $[a, b]$ and by Theorem 2.1 we get

$$(2.8) \quad 0 \leq \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \leq \frac{1}{4}f'_+(b)(b-a)$$

which gives a counterpart for the first membership of Hadamard's inequality.

3 A QUADRATURE FORMULA OF RIEMANN TYPE

Let $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$ and $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, \dots, n-1$) a sequence of intermediate points for I_n . Construct the Riemann sums

$$R_n(f, I_n, \xi) = \sum_{i=0}^{n-1} f(\xi_i) h_i$$

where $h_i := x_{i+1} - x_i$.

We have the following quadrature formula

Theorem 3.1. *Let $f : [a, b] \rightarrow \mathbf{R}$ be an L -lipschitzian mapping on $[a, b]$ and I_n, ξ_i ($i = 0, \dots, n-1$) be as above. Then we have the Riemann quadrature formula*

$$(3.1) \quad \int_a^b f(x) dx = R_n(f, I_n, \xi) + W_n(f, I_n, \xi)$$

where the remainder satisfies the estimation

$$(3.2) \quad |W_n(f, I_n, \xi)| \leq \frac{1}{4}L \sum_{i=0}^{n-1} h_i^2 + L \sum_{i=0}^{n-1} \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2$$

$$\leq \frac{1}{2}L \sum_{i=0}^{n-1} h_i^2$$

for all ξ_i ($i = 0, \dots, n-1$) as above.

The constant $\frac{1}{4}$ is sharp in (3.2).

Proof. Apply Theorem 2.1 on the interval $[x_i, x_{i+1}]$ to get

$$(3.3) \quad \left| \int_{x_i}^{x_{i+1}} f(x) dx - f(\xi_i) h_i \right| \leq L \left[\frac{1}{4} h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right].$$

Summing over i from 0 to $n - 1$ and using the generalized triangle inequality we get

$$\begin{aligned} |W_n(f, I_n, \xi)| &\leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x) dx - f(\xi_i) h_i \right| \\ &\leq L \sum_{i=0}^{n-1} \left[\frac{1}{4} h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right]. \end{aligned}$$

Now, as

$$\left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \leq \frac{1}{4} h_i^2$$

for all $\xi_i \in [x_i, x_{i+1}] (i = 0, \dots, n - 1)$ the second part of (3.2) is also proved. ■

Note that the best estimation we can get from (3.2) is that one for which $\xi_i = \frac{x_i + x_{i+1}}{2}$ obtaining the following midpoint formula:

Corollary 3.2. *Let f, I_n be as above. Then we have the midpoint rule*

$$\int_a^b f(x) dx = M_n(f, I_n) + S_n(f, I_n)$$

where

$$M_n(f, I_n) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i$$

and the remainder $S_n(f, I_n)$ satisfies the estimation

$$|S_n(f, I_n)| \leq \frac{1}{4} L \sum_{i=0}^{n-1} h_i^2.$$

Remark 3.1. *If we assume that $f : [a, b] \rightarrow \mathbf{R}$ is differentiable on (a, b) and whose derivative f' is bounded on (a, b) we can put instead of L the infinity norm $\|f'\|_\infty$ obtaining the estimation due to Dragomir-Wang from the paper [1].*

4 APPLICATIONS FOR EULER'S BETA MAPPING

Consider the mapping Beta for real numbers

$$B(p, q) := \int_0^1 t^{p-1} (1-t)^{q-1} dt, \quad p, q > 0$$

and the mapping $e_{p,q}(t) := t^{p-1} (1-t)^{q-1}, t \in [0, 1]$.

We have for $p, q > 1$ that

$$e'_{p,q}(t) = e_{p-1,q-1}(t) [p-1 - (p+q-2)t].$$

If $t \in \left[0, \frac{p-1}{p+q-2}\right)$ then $e'_{p,q}(t) > 0$ and if $t \in \left(\frac{p-1}{p+q-2}, 1\right]$ then $e'_{p,q}(t) < 0$ which shows that for $t_0 = \frac{p-1}{p+q-2}$ we have a maximum for $e_{p,q}$ and then:

$$\sup_{t \in [0,1]} e_{p,q}(t) = e_{p,q}(t_0) = \frac{(p-1)^{p-1} (q-1)^{q-1}}{(p+q-2)^{p+q-2}}; \quad p, q > 1.$$

Consequently

$$\begin{aligned} |e'_{p,q}(t)| &\leq \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \max_{t \in [0,1]} |p-1-(p+q-2)t| \\ &= \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}}; \quad p, q > 2 \end{aligned}$$

for all $t \in [0, 1]$ and then

$$(4.1) \quad \|e'_{p,q}\|_{\infty} \leq \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \quad p, q > 2.$$

The following inequality for Beta mapping holds

Proposition 4.1. *Let $p, q > 2$ and $x \in [0, 1]$. Then we have the inequality*

$$(4.2) \quad \begin{aligned} &|B(p, q) - x^{p-1}(1-x)^{q-1}| \\ &\leq \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \left[\frac{1}{4} + \left(x - \frac{1}{2}\right)^2 \right]. \end{aligned}$$

The proof follows by Theorem 2.1 applied for the mapping $e_{p,q}$ and taking into account that $\|e'_{p,q}\|_{\infty}$ satisfies the inequality (4.1).

Corollary 4.2. *Let $p, q > 2$. Then we have the inequality*

$$\left| B(p, q) - \frac{1}{2^{p+q-2}} \right| \leq \frac{1}{4} \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}}.$$

Now, if we apply Theorem 3.1 for the mapping $e_{p,q}$ we get the following approximation of Beta mapping in terms of Riemann sums.

Proposition 4.3. *Let $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$, $\xi_i \in [x_i, x_{i+1}]$, $(i = 0, \dots, n-1)$ a sequence of intermediate points for I_n and $p, q > 2$. Then we have the formula*

$$B(p, q) = \sum_{i=0}^{n-1} \xi_i^{p-1} (1 - \xi_i)^{q-1} h_i + T_n(p, q)$$

where the remainder $T_n(p, q)$ satisfies the estimation

$$\begin{aligned} |T_n(p, q)| &\leq \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \\ &\quad \times \left[\frac{1}{4} \sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1} \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \\ &\leq \frac{1}{2} \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \sum_{i=0}^{n-1} h_i^2. \end{aligned}$$

Particularly, if we choose above $\xi_i = \frac{x_i + x_{i+1}}{2}$ ($i = 0, \dots, n-1$) then we get the approximation

$$B(p, q) = \frac{1}{2^{p+q-2}} \sum_{i=0}^{n-1} (x_i + x_{i+1})^{p-1} (2 - x_i - x_{i+1})^{q-1} + V_n(p, q)$$

where

$$|V_n(p, q)| \leq \frac{1}{4} \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \sum_{i=0}^{n-1} h_i^2.$$

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