

## AN INEQUALITY OF OSTROWSKI TYPE AND ITS APPLICATIONS FOR SIMPSON'S RULE AND SPECIAL MEANS

I. Fedotov and S. S. Dragomir

ABSTRACT. An integral inequality of Ostrowski type and its applications for special means and error evaluation in Simpson's quadrature rule are given.

### 1 INTRODUCTION

The following integral inequality which establishes a connection between the integral of the product and the product of the integrals is well known in literature as Grüss' inequality [2, p. 296].

**Theorem 1.1.** *Let  $f, g : [a, b] \rightarrow \mathbf{R}$  be two integrable functions such that  $\varphi \leq f(x) \leq \Phi$  and  $\gamma \leq g(x) \leq \Gamma$  for all  $x \in [a, b]$ ;  $\varphi, \Phi, \gamma$  and  $\Gamma$  are constants. Then we have the inequality*

$$(1.1) \quad \left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right| \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma)$$

and the inequality is sharp in the sense that the constant  $\frac{1}{4}$  can not be replaced by a smaller one.

In 1938, Ostrowski (cf., for example [3, p. 468]), proved the following inequality which gives an approximation of the integral  $\frac{1}{b-a} \int_a^b f(t)dt$  as follows:

**Theorem 1.2.** *Let  $f : [a, b] \rightarrow \mathbf{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  whose derivative  $f' : (a, b) \rightarrow \mathbf{R}$  is bounded on  $(a, b)$ , i.e.,  $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$ . Then:*

$$(1.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for all  $x \in [a, b]$ . The constant  $\frac{1}{4}$  is best.

In the recent paper [1], S.S. Dragomir and S. Wang have proved the following version of Ostrowski's inequality by using Grüss' inequality (1.1).

---

Date. January, 1999.

1991 Mathematics Subject Classification. Primary 26 D 15; Secondary 41 A 55.

Key words and phrases. Ostrowski Inequality, Simpson's Rule.

**Theorem 1.3.** Let  $f : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$  be a differentiable mapping in the interior of  $I$  and let  $a, b \in \text{int}(I)$  with  $a < b$ . If  $f' \in L_1[a, b]$  and

$$\gamma \leq f'(x) \leq \Gamma \quad \text{for all } x \in [a, b],$$

then we have the following inequality:

$$(1.3) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(b) - f(a)}{b-a} \left( x - \frac{a+b}{2} \right) \right| \leq \frac{1}{4} (b-a) (\Gamma - \gamma)$$

for all  $x \in [a, b]$ .

They also applied this result for special means and in Numerical Integration obtaining some quadrature formulae generalizing the mid-point quadrature rule and the trapezoid rule. Note that the error bounds they obtained are in terms of the first derivative which are particularly useful in the case when  $f''$  does not exist or is very large at some points in  $[a, b]$ .

In this paper, we give a generalization of the above inequality which contains in a particular case the classical *Simpson formula*. Application for special means and in *Numerical Integration* are also given.

## 2 AN INTEGRAL INEQUALITY OF GRÜSS TYPE

For any real numbers  $a < b$ , let us consider the function

$$p(t) \equiv p_x(t) = \begin{cases} t - a + A, & \text{if } a \leq t \leq x \\ t - b + B & \text{if } x < t \leq b. \end{cases}$$

It is clear that  $p_x$  has the following properties.

(a) It has the jump

$$[p]_x = (B - A) - (b - a) \quad \text{at the point } t = x$$

and

$$\frac{dp_x(t)}{dt} = 1 + [p]_x \delta(t - x).$$

(b) Let  $M_x := \sup_{t \in (a,b)} p_x(t)$  and  $m_x := \inf_{t \in (a,b)} p_x(t)$ . Then the difference  $M_x - m_x$  can be evaluated as follows :

1. For  $B - A \leq 0$  we have

$$M_x - m_x = -[p]_x.$$

2. For  $B - A > 0$ , the following three cases are possible

(i) If  $0 \leq B - A \leq \frac{1}{2}(b - a)$ , then

$$M_x - m_x = \begin{cases} -x + b & \text{for } a \leq x \leq a + (B - A); \\ -[p]_x & \text{for } a + (B - A) < x \leq b - (B - A); \\ x - a & \text{for } b - (B - A) < x \leq b. \end{cases}$$

(ii) If  $\frac{1}{2}(b-a) < B-A \leq (b-a)$ , then

$$M_x - m_x = \begin{cases} -x + b & \text{for } a \leq x < b - (B - A); \\ B - A & \text{for } b - (B - A) \leq x < a + (B - A); \\ x - a & \text{for } a + (B - A) \leq x \leq b. \end{cases}$$

(iii) If  $B - A > b - a$ , then

$$M_x - m_x = B - A.$$

The following inequality of Ostrowski type holds.

**Theorem 2.1.** Let  $f : [a, b] \rightarrow \mathbf{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  whose derivative satisfies the assumption

$$(2.1) \quad \gamma \leq f'(t) \leq \Gamma \quad \text{for all } t \in (a, b),$$

where  $\gamma, \Gamma$  are given real numbers. Then we have the inequality:

$$(2.2) \quad \left| (C - A)f(a) + (b - a - B + A)f(x) + (B - C)f(b) - \int_a^b f(t) dt \right| \leq \frac{1}{4}(\Gamma - \gamma)(M_x - m_x)(b - a)$$

where

$$C = C(x) := \frac{1}{2(b-a)} [(x-a)(x-a+2A) - (x-b)(x-b+2B)],$$

and  $A, B, M_x$  and  $m_x$  are as above,  $x \in [a, b]$ .

*Proof.* Using Grüss' inequality (1.1) we can state that

$$(2.3) \quad \left| \frac{1}{b-a} \int_a^b p_x(t) f'(t) dt - \frac{f(b) - f(a)}{b-a} \cdot \frac{1}{b-a} \int_a^b p_x(t) dt \right| \leq \frac{1}{4}(\Gamma - \gamma)(M_x - m_x),$$

for all  $x \in [a, b]$ .

Integrating first term, by parts, we obtain:

$$\int_a^b p_x(t) f'(t) dt = Bf(b) - Af(a) - \int_a^b f(t) dt - [p]_x f(x).$$

Also, as

$$\int_a^b p_x(t) dt = \frac{1}{2} [(x-a)(x-a+2A) - (x-b)(x-b+2B)]$$

then (2.3) gives the inequality:

$$\left| \frac{1}{b-a} \left[ Bf(b) - Af(a) - \int_a^b f(t) dt - [p]_x f(x) \right] - C \cdot \frac{f(b) - f(a)}{b-a} \right| \leq \frac{1}{4} (\Gamma - \gamma) (M_x - m_x)$$

which is clearly equivalent with the desired result (2.2). ■

**Remark 2.1.** Setting in (2.2)  $A = B = 0$  and taking into account, by property (b), that  $M_x - m_x = b - a$ , we obtain the inequality (1.3) by Dragomir and Wang.

The following corollary is interesting:

**Corollary 2.2.** Let  $A, B$  real numbers so that  $0 \leq B - A \leq \frac{(b-a)}{2}$ . If  $f$  is as above, then we have the inequality

$$(2.4) \quad \left| \frac{B-A}{2} f(a) + [b-a - (B-A)] f\left(\frac{a+b}{2}\right) + \frac{B-A}{2} f(b) - \int_a^b f(t) dt \right| \leq \frac{1}{4} (\Gamma - \gamma) (b-a - B + A) (b-a).$$

*Proof.* Consider  $x = \frac{a+b}{2}$ . Then

$$x - a = \frac{b-a}{2}, \quad x - b = -\frac{b-a}{2}$$

$$C = \frac{A+B}{2}, \quad x \in [a + (B-A), b - (B-A)].$$

By property (b) we have

$$M_x - m_x = (b-a) - (B-A).$$

Applying Theorem 2.1 for  $x = \frac{a+b}{2}$ , we get easily (2.4). ■

**Remark 2.2.** 1. If we choose in the above corollary  $B - A = \frac{b-a}{2}$ , then we get

$$(2.5) \quad \left| \frac{1}{2} \left[ \frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] (b-a) - \int_a^b f(t) dt \right| \leq \frac{1}{8} (\Gamma - \gamma) (b-a)^2$$

which is a combination between mid-point and trapezoid formula.

2. If we choose in (2.4),  $B = A$ , then we get the mid-point inequality

$$(2.6) \quad \left| (b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \leq \frac{1}{4} (\Gamma - \gamma) (b-a)^2$$

proved by S.S. Dragomir and S. Wang in [1] (Corollary 2.3).

3. If we choose in (2.4),  $B - A = \frac{b-a}{3}$ , then we obtain Simpson's formula

$$(2.7) \quad \left| \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \int_a^b f(t) dt \right| \\ \leq \frac{1}{6} (\Gamma - \gamma) (b-a)^2$$

for which we have an estimation in terms of the first derivative not as in the classical case in which the fourth derivative is required, i.e.,

$$(2.8) \quad \left| \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{s}\right) + f(b) \right] - \int_a^b f(t) dt \right| \\ \leq \frac{\|f^{(4)}\|_{\infty}}{2880} (b-a)^5.$$

**Remark 2.3.** The method of evaluation of the error for the Simpson rule considered above can be applied for any quadrature formula of Newton-Cotes type.

For example, to get the similar evaluation of the error for the Newton-Cotes rule of order 3, it is sufficient to replace the function  $p_x(t)$  in (2.3) by the function:

$$p_x(t) := \begin{cases} t - a - A & \text{if } a \leq t \leq a + h \\ t - \frac{a+b}{2} + \frac{A+B}{2} & \text{if } a + h < t \leq b - h \\ t - b - B & \text{if } b - h < t \leq b \end{cases}$$

where  $B - A = \frac{b-a}{4}$ ,  $h = \frac{b-a}{3}$ .

### 3 APPLICATION FOR SPECIAL MEANS

Let us recall some important means of positive real numbers.

(a) The arithmetic mean :

$$A = A(a, b) := \frac{a+b}{2}, \quad a, b > 0;$$

(b) The geometric mean:

$$G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0;$$

(c) The harmonic mean:

$$H = H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad a, b > 0;$$

(d) The logarithmic mean:

$$L = L(a, b) := \begin{cases} a & \text{if } a = b, \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b, \end{cases} \quad a, b > 0;$$

(e) The identric mean :

$$I = I(a, b) := \begin{cases} \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b, \\ a & \text{if } a = b, \end{cases} \quad a, b > 0;$$

(f) The  $p$ -logarithmic mean

$$L_p = L_p(a, b) := \begin{cases} \left[ \frac{b^{p+1} - a^{p+1}}{(p-1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b, \\ a & \text{if } a = b, \end{cases} \quad \begin{array}{l} a, b > 0, \\ p \in \mathbf{R} \setminus \{-1, 0\}. \end{array}$$

In what follows we shall apply the inequality (2.7) written in the following form

$$(3.1) \quad \left| \frac{2}{3} f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{6} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{6} (\Gamma - \gamma) (b-a).$$

for the previous means.

Consider the mapping  $f(x) = x^p$  ( $p > 1$ ),  $x > 0$ . Then

$$f\left(\frac{a+b}{2}\right) = A^p(a, b), \quad \frac{f(a)+f(b)}{2} = A(a^p, b^p),$$

$$\frac{1}{b-a} \int_a^b f(t) dt = L_p^p(a, b), \quad \Gamma - \gamma = (a-b)(p-1) L_{p-2}^{p-2}$$

for  $a, b \in \mathbf{R}$  with  $0 < a < b$ . Consequently, we have the inequality

$$(3.2) \quad \left| \frac{2}{3} A^p(a, b) + \frac{1}{3} A(a^p, b^p) - L_p^p(a, b) \right| \leq \frac{1}{6} (b-a)^2 (p-1) L_{p-2}^{p-2}$$

Consider the mapping  $f(x) = \frac{1}{x}$ ,  $x > 0$ . Then

$$f\left(\frac{a+b}{2}\right) = A^{-1}(a, b), \quad \frac{f(a)+f(b)}{2} = H^{-1}(a, b),$$

$$\frac{1}{b-a} \int_a^b f(t) dt = L^{-1}(a, b),$$

$$\Gamma - \gamma = \frac{b^2 - a^2}{a^2 b^2} = 2 \frac{(b-a) A(a, b)}{G^4(a, b)}$$

for  $0 < a < b$ . Consequently, we have the inequality:

$$\left| \frac{2}{3}A^{-1}(a, b) + \frac{1}{3}H^{-1}(a, b) - L^{-1}(a, b) \right| \leq \frac{1}{3}(b-a)^2 \frac{A(a, b)}{G^4(a, b)}$$

which is equivalent to

$$(3.3) \quad \left| \frac{2}{3}HL + \frac{1}{3}AL - AH \right| \leq \frac{1}{3}(b-a)^2 \frac{A^2HL}{G^4}.$$

Consider the mapping  $f(x) = \ln x, x > 0$ . Then we have

$$f\left(\frac{a+b}{2}\right) = \ln A, \quad \frac{f(a) + f(b)}{2} = \ln G,$$

$$\frac{1}{b-a} \int_a^b f(t) dt = \ln I, \quad \Gamma - \gamma = \frac{b-a}{G^2}$$

for  $a, b \in \mathbf{R}$  with  $0 < a < b$ . Consequently, we have the inequality

$$\left| \frac{2}{3} \ln A + \frac{1}{3} \ln G - \ln I \right| \leq \frac{1}{6} \frac{(b-a)^2}{G^2}$$

which is equivalent to

$$(3.4) \quad \left| \ln \left( \frac{A^{\frac{2}{3}} G^{\frac{1}{3}}}{I} \right) \right| \leq \frac{1}{6} \frac{(b-a)^2}{G^2}.$$

#### 4 A NEW ESTIMATION OF THE ERROR BOUND IN SIMPSON'S RULE

The following theorem holds.

**Theorem 4.1.** *Let  $f : [a, b] \rightarrow \mathbf{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  whose derivative satisfies the condition (2.1), i.e.,*

$$\gamma \leq f'(t) \leq \Gamma \quad \text{for all } t \in (a, b);$$

where  $\gamma, \Gamma$  are given real numbers. Then we have

$$(4.1) \quad \int_a^b f(t) dt = S_n(I_n, f) + R_n(I_n, f)$$

where

$$S_n(I_n, f) = \frac{1}{3} \sum_{i=0}^{n-1} h_i [f(x_i) + 4f(x_i + h_i) + f(x_{i+1})],$$

$I_n$  is the partition given by  $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ ,  $h_i := \frac{1}{2}(x_{i+1} - x_i)$ ,  $i = 0, \dots, n-1$ ; and the remainder term  $R_n(I_n, f)$  satisfies the estimation:

$$(4.2) \quad |R_n(I_n, f)| \leq \frac{2}{3} (\Gamma - \gamma) \sum_{i=0}^{n-1} h_i^2.$$

*Proof.* Let us set in (2.7)  $a = x_i, b = x_{i+1}, 2h_i = x_{i+1} - x_i$ , where  $i = 0, \dots, n-1$ .

Then we have the estimation:

$$\left| \frac{h_i}{3} [f(x_i) + 4f(x_i + h_i) + f(x_{i+1})] - \int_{x_i}^{x_{i+1}} f(t) dt \right| \leq \frac{2}{3} (\Gamma - r) h_i^2,$$

for all  $i = 0, \dots, n-1$ .

After summing and using the triangle inequality, we obtain

$$\left| \sum_{i=0}^{n-1} \frac{h_i}{3} [f(x_i) + 4f(x_i + h_i) + f(x_{i+1})] - \int_a^b f(t) dt \right| \leq \frac{2}{3} (\Gamma - \gamma) \sum_{i=0}^{n-1} h_i^2$$

which proves the required estimation (4.2). ■

**Corollary 4.2.** *Under the above assumptions and if we put*

$$\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty,$$

*then we have the following estimation of the remainder term in Simpson's formula*

$$(4.3) \quad |R_n(I_n, f)| \leq \frac{4}{3} \|f'\|_\infty \sum_{i=0}^{n-1} h_i^2.$$

**Remark 4.1.** *The classical error estimates based on the Taylor expansion for the Simpson's rule involve the fourth derivative  $\|f^{(4)}\|_\infty$ . In the case when  $f^{(4)}$  does not exist, or is very large at some points in  $[a, b]$ , the classical estimates can not be applied, and thus (4.2) and (4.3) provide alternative error estimates for the Simpson's rule.*

## REFERENCES

- [1] S.S. DRAGOMIR and S. WANG, An inequality of Ostrowski-Grüss' type and its applications to the estimations of error bounds for some special means and for some numerical quadrature rules, *Computing Math. Appl.* **33**(11) (1997), 15-20.
- [2] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993.
- [3] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, 1994.

(I. FEDOTOV) DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF TRANSKEI, PRIVATE BAG X1, UNITRA UMTATA, 5117, SOUTH AFRICA

(S. S. DRAGOMIR) SCHOOL OF COMMUNICATIONS AND INFORMATICS, VICTORIA UNIVERSITY OF TECHNOLOGY, PO BOX 14428, MELBOURNE CITY MC, VICTORIA 8001, AUSTRALIA.

*E-mail address:* I. Fedotov fedotov@getafix.utr.ac.za

S. S. Dragomir sever@matilda.vu.edu.au