

ON AN INEQUALITY FOR LOGARITHMS AND APPLICATIONS IN INFORMATION THEORY

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ABSTRACT. A new analytic inequality for logarithms which provides a converse to arithmetic mean-geometric mean inequality and its applications in information theory are given.

1 INTRODUCTION

The present paper continues the investigations started in [1], where the main result is

Theorem 1.1. Let $\xi_k \in (0, \infty)$, $p_k > 0$, $k = 1, \dots, n$ with $\sum_{k=1}^n p_k = 1$ and $b > 1$.

Then

$$(1.1) \quad 0 \leq \log_b \left(\sum_{k=1}^n p_k \xi_k \right) - \sum_{k=1}^n p_k \log_b \xi_k \\ \leq \frac{1}{2 \ln b} \sum_{k,i=1}^n \frac{p_k p_i}{\xi_k \xi_i} (\xi_i - \xi_k)^2.$$

The equality holds in both inequalities simultaneously if and only if $\xi_1 = \dots = \xi_n$.

2 A NEW ANALYTIC INEQUALITY FOR LOGARITHMS

We shall start to the following analytic inequality for logarithms which provides a different bound than the inequality of Dragomir-Goh (1.1):

Theorem 2.1. Let $\xi_k \in [1, \infty)$ and $p_k > 0$ with $\sum_{k=1}^n p_k = 1$ and $b > 1$.

Then we have

$$(2.1) \quad 0 \leq \log_b \left(\sum_{k=1}^n p_k \xi_k \right) - \sum_{k=1}^n p_k \log_b \xi_k \\ \leq \frac{1}{4 \ln b} \sum_{i,j=1}^n p_i p_j (\xi_i - \xi_j)^2.$$

The equality holds in both inequalities simultaneously if and only if $\xi_1 = \dots = \xi_n$.

Proof. We shall use the well known Jensen's discrete inequality for convex mappings which states that

$$(2.2) \quad f \left(\sum_{i=1}^n p_i x_i \right) \leq \sum_{i=1}^n p_i f(x_i)$$

for all $p_i > 0$, $\sum_{i=1}^n p_i = 1$, f a convex mapping on a given interval I and $x_i \in I$ ($i = 1, \dots, n$).

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Now, let consider the mapping $f : [1, \infty) \rightarrow \mathbf{R}$, $f(x) = \frac{x^2}{2} + \ln x$. Then

$$f'(x) = x + \frac{1}{x} = \frac{x^2 + 1}{x} \quad \text{for all } x \in [1, \infty)$$

and

$$f''(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2} \quad \text{for all } x \in [1, \infty),$$

i.e., f is a strictly convex mapping on $[1, \infty)$.

Applying Jensen's discrete inequality for convex mappings, we have

$$(2.3) \quad \frac{1}{2} \left(\sum_{i=1}^n p_i \xi_i \right)^2 + \ln \left(\sum_{i=1}^n p_i \xi_i \right) \leq \frac{1}{2} \sum_{i=1}^n p_i \xi_i^2 + \sum_{i=1}^n p_i \ln \xi_i$$

which is equivalent to

$$\ln \left(\sum_{i=1}^n p_i \xi_i \right) - \sum_{i=1}^n p_i \ln \xi_i \leq \frac{1}{2} \left[\sum_{i=1}^n p_i \xi_i^2 - \left(\sum_{i=1}^n p_i \xi_i \right)^2 \right].$$

But

$$\begin{aligned} \sum_{i,j=1}^n p_i p_j (\xi_i - \xi_j)^2 &= \sum_{i,j=1}^n p_i p_j [\xi_i^2 + \xi_j^2 - 2\xi_i \xi_j] \\ &= 2 \left[\sum_{i=1}^n p_i \sum_{i=1}^n p_i \xi_i^2 - \left(\sum_{i=1}^n p_i \xi_i \right)^2 \right] = 2 \left[\sum_{i=1}^n p_i \xi_i^2 - \left(\sum_{i=1}^n p_i \xi_i \right)^2 \right] \end{aligned}$$

and then the above inequality becomes

$$(2.4) \quad \ln \left(\sum_{i=1}^n p_i \xi_i \right) - \sum_{i=1}^n p_i \ln \xi_i \leq \frac{1}{4} \sum_{i,j=1}^n p_i p_j (\xi_i - \xi_j)^2.$$

Now, as $\log_b x = \frac{\ln x}{\ln b}$, the inequality (2.4) is equivalent to the desired inequality (2.1).

The case of equality follows by the strict convexity of f and we omit the details. ■

Remark 2.1. *Define*

$$B_1 := \frac{1}{2 \ln b} \sum_{i,j=1}^n \frac{p_i p_j}{\xi_i \xi_j} (\xi_i - \xi_j)^2 \quad (\text{as in Theorem 1.1})$$

and

$$B_2 := \frac{1}{4 \ln b} \sum_{i,j=1}^n p_i p_j (\xi_i - \xi_j)^2 \quad (\text{as in Theorem 2.1})$$

and compute the difference

$$\begin{aligned} B_1 - B_2 &= \frac{1}{2 \ln b} \sum_{i,j=1}^n p_i p_j (\xi_i - \xi_j)^2 \left[\frac{1}{\xi_i \xi_j} - 2 \right] \\ &= \frac{1}{4 \ln b} \sum_{i,j=1}^n \frac{p_i p_j (\xi_i - \xi_j)^2}{\xi_i \xi_j} (2 - \xi_i \xi_j). \end{aligned}$$

Consequently, if $\xi_i \in [1, \infty)$ so that $\xi_i \xi_j \leq 2$ for all $i, j \in \{1, \dots, n\}$, then the bound B_2 provided by Theorem 2.1 is better than the bound B_1 provided by Theorem 1.1. If $\xi_i \in [1, \infty)$ so that $\xi_i \xi_j \geq 2$ for all $i, j \in \{1, \dots, n\}$, then Theorem 1.1 provides a better result than Theorem 2.1.

We give now some applications of the above results for arithmetic mean-geometric mean inequality.

Recall that for $q_i > 0$ with $Q_n := \sum_{i=1}^n q_i$, the *arithmetic mean* of x_i with the weights q_i , $i \in \{1, \dots, n\}$ is

$$(A) \quad A_n(\bar{q}, \bar{x}) := \frac{1}{Q_n} \sum_{i=1}^n q_i x_i$$

and the *geometric mean* of x_i with the weights q_i , $i \in \{1, \dots, n\}$, is

$$(G) \quad G_n(\bar{q}, \bar{x}) := \left(\prod_{i=1}^n x_i^{q_i} \right)^{\frac{1}{Q_n}}.$$

It is well known that the following inequality so called *arithmetic mean-geometric mean* inequality, holds

$$(2.5) \quad A_n(\bar{q}, \bar{x}) \geq G_n(\bar{q}, \bar{x})$$

with equality if and only if $x_1 = \dots = x_n$.

Now, using Theorem 1.1, we can state the following proposition containing a counterpart of the arithmetic mean-geometric mean inequality (2.5):

Proposition 2.2. *With the above assumptions for \bar{q} and \bar{x} , we have*

$$(2.6) \quad 1 \leq \frac{A_n(\bar{q}, \bar{x})}{G_n(\bar{q}, \bar{x})} \leq \exp_b \left[\frac{1}{2Q_n^2 \ln b} \sum_{i,j=1}^n \frac{q_i q_j}{x_i x_j} (x_i - x_j)^2 \right]$$

where $\exp_b(x) = b^x$, ($b > 1$). The equality holds in both inequalities simultaneously if and only if $x_1 = \dots = x_n$.

Also, using Theorem 2.1, we have another converse inequality for (2.5).

Proposition 2.3. *Let \bar{q} be as above and $\bar{x} \in \mathbf{R}^n$, so that $x_i \geq 1$, $i = 1, \dots, n$. Then we have the inequality:*

$$(2.7) \quad 1 \leq \frac{A_n(\bar{q}, \bar{x})}{G_n(\bar{q}, \bar{x})} \leq \exp_b \left[\frac{1}{4Q_n^2 \ln b} \sum_{i,j=1}^n \frac{q_i q_j}{x_i x_j} (x_i - x_j)^2 \right]$$

where $b > 1$. The equality holds in both inequalities simultaneously if and only if $x_1 = \dots = x_n$.

Remark 2.2. *As in the previous remark, if $1 \leq x_i x_j \leq 2$ then the bound (2.7) is better than (2.6). If $x_i x_j \geq 2$, then (2.6) is better than (2.7).*

3 APPLICATIONS FOR THE ENTROPY MAPPING

Let us consider now, the *b-entropy mapping* of the discrete random variable X with n possible outcomes and having the probability distribution $p = (p_i)$, $i = \{1, \dots, n\}$:

$$H_b(X) = \sum_{i=1}^n p_i \log_b \left(\frac{1}{p_i} \right).$$

We know (see [1]) that the following counterpart inequality holds:

$$(3.1) \quad 0 \leq \log_b n - H_b(X) \leq \frac{1}{2 \ln b} \sum_{i,j=1}^n (p_i - p_j)^2$$

with equality if and only if $p_i = \frac{1}{n}$ for all $i \in \{1, \dots, n\}$.

The following similar result also holds:

Theorem 3.1. *Let X be as above. Then we have*

$$(3.2) \quad 0 \leq \log_b n - H_b(X) \leq \frac{1}{4 \ln b} \sum_{i,j=1}^n \frac{(p_i - p_j)^2}{p_i p_j}.$$

The equality holds if and only if $p_i = \frac{1}{n}$ for all $i \in \{1, \dots, n\}$.

Proof. As $p_i \in (0, 1]$, then $\xi_i = \frac{1}{p_i} \in [1, \infty)$ and we can apply Theorem 2.1 to get

$$\begin{aligned} 0 \leq \log_b n - H_b(X) &\leq \frac{1}{4 \ln b} \sum_{i,j=1}^n p_i p_j \left(\frac{1}{p_i} - \frac{1}{p_j} \right)^2 \\ &= \frac{1}{4 \ln b} \sum_{i,j=1}^n \frac{(p_i - p_j)^2}{p_i p_j}. \end{aligned}$$

The equality holds iff $\xi_i = \xi_j$ for all $i, j \in \{1, \dots, n\}$ which is equivalent to $p_i = p_j$ for all $i, j \in \{1, \dots, n\}$, i.e., $p_i = \frac{1}{n}$ for all $i \in \{1, \dots, n\}$. ■

The following corollary is important in applications as it provides a sufficient condition for the probability p so that $\log_b n - H_b(X)$ is small enough.

Corollary 3.2. *Let X be as above and $\varepsilon > 0$. If the probabilities $p_i, i = 1, \dots, n$, satisfy the conditions:*

$$(3.3) \quad \frac{1}{2} \left[2 + k - \sqrt{k(k+4)} \right] \leq \frac{p_i}{p_j} \leq \frac{1}{2} \left[2 + k + \sqrt{k(k+4)} \right]$$

for all $1 \leq i < j \leq n$, where

$$k = \frac{4\varepsilon \ln b}{n(n-1)} \quad (n \geq 2),$$

then we have the estimation

$$(3.4) \quad 0 \leq \log_b n - H_b(X) \leq \varepsilon.$$

Proof. Let observe that

$$\frac{1}{4 \ln b} \sum_{i,j=1}^n \frac{(p_i - p_j)^2}{p_i p_j} = \frac{1}{2 \ln b} \sum_{1 \leq i < j \leq n} \frac{(p_i - p_j)^2}{p_i p_j}.$$

Suppose that

$$\frac{(p_i - p_j)^2}{p_i p_j} \leq k, \quad \text{for } 1 \leq i < j \leq n.$$

Then

$$p_i^2 - (2+k)p_i p_j + p_j^2 \leq 0 \quad \text{for } 1 \leq i < j \leq n.$$

Denoting $t = \frac{p_i}{p_j}$, the above inequality is equivalent to $t^2 - (2+k)t + 1 \leq 0$, i.e., $t \in [t_1, t_2]$, where

$$t_1 = \frac{2+k-\sqrt{k(k+4)}}{2} \quad \text{and} \quad t_2 = \frac{2+k+\sqrt{k(k+4)}}{2}.$$

If we choose $k = \frac{4\varepsilon \ln b}{n(n-1)}$, then by (3.2) we have

$$\begin{aligned} 0 &\leq \log_b n - H_b(X) \leq \frac{1}{4 \ln b} \sum_{i,j=1}^n \frac{(p_i - p_j)^2}{p_i p_j} \\ &= \frac{1}{2 \ln b} \sum_{1 \leq i < j \leq n} \frac{(p_i - p_j)^2}{p_i p_j} \\ &\leq \frac{1}{2 \ln b} \sum_{1 \leq i < j \leq n} k = \frac{n(n-1)}{4 \ln b} \cdot \frac{4\varepsilon \ln b}{n(n-1)} = \varepsilon, \end{aligned}$$

and the corollary is proved. ■

Now, consider the bounds

$$M_1 := \frac{1}{2 \ln b} \sum_{i,j=1}^n (p_i - p_j)^2 \quad (\text{given by (3.1)})$$

and

$$M_2 := \frac{1}{4 \ln b} \sum_{i,j=1}^n \frac{(p_i - p_j)^2}{p_i p_j} \quad (\text{given by (3.2)}).$$

We give an example for which M_1 is less than M_2 and another example for which M_2 is less than M_1 which will suggest that we can use both of them to estimate the above difference $\log_b n - H_b(X)$.

Consider the probability distribution:

$$p_1 = 0.3475, \quad p_2 = 0.2398, \quad p_3 = 0.1654$$

$$p_4 = 0.1142, \quad p_5 = 0.0788, \quad p_6 = 0.0544.$$

In this case

$$\bar{M}_1 = 6.5119, \quad \bar{M}_2 = 12.1166,$$

where

$$\bar{M}_1 := \frac{1}{2} \sum_{i,j=1}^n (p_i - p_j)^2, \quad \bar{M}_2 := \frac{1}{4} \sum_{i,j=1}^n \frac{(p_i - p_j)^2}{p_i p_j} \quad \text{and } n = 6.$$

Consider the probability distribution

$$p_1 = 0.2468, \quad p_2 = 0.2072, \quad p_3 = 0.1740$$

$$p_4 = 0.1461, \quad p_5 = 0.1227, \quad p_6 = 0.1031.$$

In this case,

$$\bar{M}_1 = 5.2095, \quad \bar{M}_2 = 2.3706.$$

4 BOUNDS FOR JOINT ENTROPY

Consider the joint entropy of two random variable X and Y [3, p. 25]:

$$H_b(X, Y) := \sum_{x,y} p(x, y) \log_b \frac{1}{p(x, y)}$$

where the joint probability $p(x, y) = P\{X = x, Y = y\}$.

In paper [2], S.S. Dragomir and C. J. Goh have proved the following result using Theorem 1.1:

Theorem 4.1. *With the above assumptions, we have that*

$$(4.1) \quad 0 \leq \log_b(rs) - H_b(X, Y) \leq \frac{1}{2 \ln b} \sum_{x,y} \sum_{u,v} (p(x, y) - p(u, v))^2$$

where the range of X contains r elements and the range of Y contains s elements. Equality holds in both inequalities simultaneously if and only if $p(x, y) = p(u, v)$ for all $(x, y), (u, v)$.

The following corollary is useful in practice:

Corollary 4.2. *With the above assumptions and if*

$$\max_{(x,y),(u,v)} |p(x, y) - p(u, v)| \leq \sqrt{\frac{2\varepsilon \ln b}{rs}}, \quad \varepsilon > 0$$

then we have the estimation

$$0 \leq \log_b(rs) - H_b(X, Y) \leq \varepsilon.$$

Now, using the second converse inequality embodied in Theorem 2.1, we are able to prove another upper bound for the difference $\log_b(rs) - H_b(X, Y)$.

Theorem 4.3. *With the above assumptions, we have*

$$(4.2) \quad 0 \leq \log_b(rs) - H_b(X, Y) \leq \frac{1}{4 \ln b} \sum_{x,y} \sum_{u,v} \frac{(p(x,y) - p(u,v))^2}{p(x,y)p(u,v)}$$

where the range of X and Y are as above. Equality holds in both inequalities simultaneously iff $p(x, y) = p(u, v)$ for all (x, y) and (u, v) .

Proof. Using Theorem 2.1, we have for $p_i = p(x, y)$ and $\xi_i = \frac{1}{p(x,y)}$,

$$\begin{aligned} 0 &\leq \log_b \left(\sum_{x,y} p(x,y) \cdot \frac{1}{p(x,y)} \right) - \sum_{x,y} p(x,y) \log_b \frac{1}{p(x,y)} \\ &\leq \frac{1}{4 \ln b} \sum_{x,y} \sum_{u,v} p(x,y) p(u,v) \left(\frac{1}{p(x,y)} - \frac{1}{p(u,v)} \right)^2 \\ &= \frac{1}{4 \ln b} \sum_{x,y} \sum_{u,v} \frac{(p(x,y) - p(u,v))^2}{p(x,y)p(u,v)} \end{aligned}$$

which is clearly equivalent to the desired result. The case of equality is obvious by Theorem 2.1. ■

The following corollary is important in practical applications:

Corollary 4.4. *Let X and Y be as above and $\varepsilon > 0$. Denote $P = \max p(x, y)$ and $p = \min p(x, y)$. If*

$$(4.3) \quad \frac{P}{p} \leq 1 + k + \sqrt{k(k+2)}$$

where

$$k := \frac{2\varepsilon \ln b}{(rs)^2},$$

then we have the bound

$$0 \leq \log_b(rs) - H_b(X, Y) \leq \varepsilon.$$

Proof. At the beginning, let us consider the inequality

$$\frac{(a-b)^2}{2ab} \leq k, \quad \text{for } a, b > 0 \text{ and } k \geq 0.$$

This inequality is clearly equivalent to

$$a^2 - 2(1+k)ab + b^2 \leq 0$$

or, denoting $t := \frac{a}{b}$, to

$$t^2 - 2(1+k)t + 1 \leq 0,$$

i.e.,

$$1 + k - \sqrt{k(k+2)} \leq t \leq 1 + k + \sqrt{k(k+2)}.$$

Now, let suppose that

$$(4.4) \quad 1 + k - \sqrt{k(k+2)} \leq \frac{p(x,y)}{p(u,v)} \leq 1 + k + \sqrt{k(k+2)}$$

for all (x, y) and (u, v) and $k := \frac{2\varepsilon \ln b}{(rs)^2}$. Then by (4.2), we have

$$\begin{aligned} 0 \leq \log_b(rs) - H_b(X, Y) &\leq \frac{1}{4 \ln b} \sum_{x, y} \sum_{u, v} \frac{(p(x, y) - p(u, v))^2}{p(x, y)p(u, v)} \\ &\leq \frac{1}{2 \ln b} \cdot (rs)^2 k = \frac{(rs)^2}{2 \ln b} \cdot \frac{2\varepsilon \ln b}{(rs)^2} = \varepsilon. \end{aligned}$$

Now, let observe that the inequality (4.4) is equivalent to:

$$1 + k - \sqrt{k(k+2)} \leq \frac{p}{P} \leq \frac{P}{p} \leq 1 + k + \sqrt{k(k+2)}.$$

But $\frac{p}{P} \geq 1 + k - \sqrt{k(k+2)}$ is equivalent to

$$\frac{P}{p} \leq \frac{1}{1 + k - \sqrt{k(k+2)}} = k + 1 + \sqrt{k(k+2)}$$

and the corollary is proved. ■

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