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ON THE OSTROWSKI'S INTEGRAL INEQUALITY FOR MAPPINGS WITH BOUNDED VARIATION AND APPLICATIONS

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Abstract. A generalization of Ostrowski's inequality for mappings with bounded variation and applications in Numerical Analysis for Euler's Beta function is given.

1 Introduction

The following theorem contains the integral inequality which is known in the literature as Ostrowski's inequality [2, p. 469].

Theorem 1.1. *Let $f : [a, b] \to \mathbf{R}$ be a differentiable mapping on (a, b) whose derivative is bounded on (a, b) and denote $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then for all $x \in [a, b]$ we have the inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty .$$

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

In this paper we prove an Ostrowski's type inequality for mappings with bounded variation and apply it in obtaining a Riemann's type quadrature formula for this class of mappings. Applications for Euler's Beta function are also given.

2 Ostrowski's Inequality for Mappings With Bounded Variation

The following inequality for mappings with bounded variation holds:

Theorem 2.1. *Let $u : [a, b] \to \mathbf{R}$ be mapping with bounded variation on $[a, b]$. Then for all $x \in [a, b]$, we have the inequality*

$$\left| \int_a^b u(t)dt - u(x)(b-a) \right| \le \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (u). \tag{2.1}$$

where $\bigvee_a^b (u)$ denotes the total variation of u.
The constant $\frac{1}{2}$ is the best possible one.

Proof. Using the integration by parts formula for Riemann-Stieltjes integral we have

$$\int_a^x (t-a)du(t) = u(x)(x-a) - \int_a^x u(t)dt$$

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and

$$\int_x^b (t-b)du(t) = u(x)(b-x) - \int_x^b u(t)dt.$$

If we add the above two equalities, we get

$$u(x)(b-a) - \int_a^b u(t)dt = \int_a^b p(x,t)du(t) \tag{2.2}$$

where

$$p(x,t) := \begin{cases} t-a & \text{if} \quad t \in [a,x) \\ \\ t-b & \text{if} \quad x \in [x,b], \end{cases}$$

for all $x,t \in [a,b]$.

Now, assume that $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \to \infty$, where $\nu(\Delta_n) := \max_{i\in\{0,...,n-1\}} (x_{i+1}^{(n)} - x_i^{(n)})$ and $\xi_i^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)}\right]$.
If $p : [a,b] \to \mathbf{R}$ is continuous on $[a,b]$ and $v : [a,b] \to \mathbf{R}$ is with bounded variation on $[a,b]$, then

$$\left|\int_a^b p(x)dv(x)\right| = \left|\lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} p(\xi_i^{(n)}) \left[v\left(x_{i+1}^{(n)}) - v(x_i^{(n)}\right)\right]\right| \tag{2.3}$$

$$\leq \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left|p(\xi_i^{(n)})\right| \left|v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right|$$

$$\leq \sup_{x\in[a,b]} |p(x)| \sup_{\Delta_n} \sum_{i=0}^{n-1} \left|v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right| = \sup_{x\in[a,b]} |p(x)| \bigvee_a^b (v).$$

Applying the inequality (2.3) for $p(x,t)$ as above and $v(x) = u(x), x \in [a,b]$, we get

$$\left|\int_a^b p(x,t)du(t)\right| \leq \sup_{t\in[a,b]} |p(x,t)| \bigvee_a^b (u) \tag{2.4}$$

$$= \max\{x-a, b-x\} \bigvee_a^b (u) = \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (u)$$

and then by (2.4), via the identity (2.2), we deduce the desired inequality (2.1).

Now, assume that the inequality (2.1) holds with a constant $C > 0$, i.e.,

$$\left|\int_a^b u(t)dt - u(x)(b-a)\right| \leq \left[C(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (u). \tag{2.5}$$

for all $x \in [a,b]$.

Consider the mapping $u : [a,b] \to \mathbf{R}$, given by

$$u(x) = \begin{cases} 0 & \text{if} \quad x \in [a,b] \backslash \left\{\frac{a+b}{2}\right\} \\ \\ 1 & \text{if} \quad x = \frac{a+b}{2} \end{cases}$$

in (2.5). Then u is with bounded variation on $[a,b]$, and

$$\bigvee_a^b(u) = 2, \qquad \int_a^b u(t)dt = 0$$

and for $x = \frac{a+b}{2}$, we get in (2.5)

$$1 \le 2C$$

which implies that $C \ge \frac{1}{2}$ and the theorem is completely proved.

The following corollary holds:

Corollary 2.2. *Let $u : [a,b] \to \mathbf{R}$ be a monotonous mapping on $[a,b]$. Then we have the inequality*

$$\left| \int_a^b u(t)dt - u(x)(b-a) \right| \le \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] |f(b) - f(a)| .$$

The case of lipschitzian mappings is embodied in the following corollary.

Corollary 2.3. *Let $u : [a,b] \to \mathbf{R}$ be an L-lipschitzian mapping on $[a,b]$, i.e., we recall*

$$|u(x) - u(y)| \le L|x-y| \quad \text{for all } x,y \in [a,b].$$

Then we have the inequality

$$\left| \int_a^b u(t)dt - u(x)(b-a) \right| \le L \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] (b-a).$$

The best inequality we can get from (2.1) is that one for which $x = \frac{a+b}{2}$ obtaining

Corollary 2.4. *Let $u : [a,b] \to \mathbf{R}$ be as above. Then we have the inequality:*

$$\left| \int_a^b u(t)dx - u\left(\frac{a+b}{2}\right)(b-a) \right| \le \frac{1}{2}(b-a) \bigvee_a^b(u). \tag{2.6}$$

Similar inequalities can be found if we assume that u is monotonous or lipschitzian on $[a,b]$. We shall omit the details.

Remark 2.1. *If we assume that u is continuous differentiable on (a,b) and u' is integrable on (a,b), then by (2.1) we get*

$$\left| \int_a^b u(t)dx - u\left(\frac{a+b}{2}\right)(b-a) \right| \le \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \|u'\|_1$$

which is the inequality obtained by Dragomir and Wang in the recent paper [1].

Remark 2.2. *It is well known that if $f : [a,b] \to \mathbf{R}$ is a convex mapping on $[a,b]$, then Hermite-Hadamard's inequality holds*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{2.7}$$

Now, if we assume that $f : I \subset \mathbf{R} \to \mathbf{R}$ is convex on I and $a,b \in Int(I), a < b$; then f'_+ is monotonous nondecreasing on $[a,b]$ and by Corollary 2.4 we get

$$0 \le \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \le \frac{1}{2} \|f'_+\|_1 \tag{2.8}$$

which gives a counterpart for the first membership of Hadamard's inequality.

Similar results can be obtained if we assume that f is convex and monotonous or convex and lipschitzian on $[a,b]$.

3 A Quadrature Formula of Riemann Type

Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ a sequence of intermediate points for I_n. Construct the Riemann sums

$$R_n(f, I_n, \boldsymbol{\xi}) = \sum_{i=0}^{n-1} f(\xi_i) h_i$$

where $h_i := x_{i+1} - x_i$.

We have the following quadrature formula

Theorem 3.1. *Let $f : [a,b] \to \mathbf{R}$ be a mapping with bounded variation on $[a,b]$ and I_n, ξ_i $(i = 0, ..., n-1)$ be as above. Then we have the Riemann quadrature formula*

$$\int_a^b f(x)dx = R_n(f, I_n, \boldsymbol{\xi}) + W_n(f, I_n, \boldsymbol{\xi}) \tag{3.1}$$

where the remainder satisfies the estimation

$$|W_n(f, I_n, \boldsymbol{\xi})| \le \sup_{i=0,\dots,n} \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_a^b(f)$$

$$\le \left[\frac{1}{2}\nu(h) + \sup_{i=0,\dots,n} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_a^b(f) \le \nu(h) \bigvee_a^b(f) \tag{3.2}$$

for all $\xi_i (i = 0, ..., n-1)$ as above, where $\nu(h) := \max_{i=0,\dots,n} h_i$.
The constant $\frac{1}{2}$ is sharp in (3.2).

Proof. Apply Theorem 2.1 on the interval $[x_i, x_{i+1}]$ to get

$$\left|\int_{x_i}^{x_{i+1}} f(x)dx - f(\xi_i)h_i\right| \le \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_{x_i}^{x_{i+1}}(f). \tag{3.3}$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality we get

$$|W_n(f, I_n, \boldsymbol{\xi})| \le \sum_{i=0}^{n-1} \left|\int_{x_i}^{x_{i+1}} f(x)dx - f(\xi_i)h_i\right|$$

$$\le \sum_{i=0}^{n-1} \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_{x_i}^{x_{i+1}}(f)$$

$$\le \sup_{i=0,\dots,n} \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}}(f).$$

$$= \sup_{i=0,\dots,n} \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_a^b(f)$$

The second inequality follows by the properties of sup(.).

Now, as

$$\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \le \frac{1}{2}h_i$$

for all $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ the last part of (3.2) is also proved.

Corollary 3.2. *Let $u : [a,b] \to \mathbf{R}$ be a monotonous mapping on $[a,b]$ and $I_n, \xi_i\ (i = 0, ..., n-1)$ be as above. Then we have the Riemann quadrature formula* (3.1) *and the remainder satisfies the estimation*

$$|W_n(f, I_n, \boldsymbol{\xi})| \le \sup_{i=0,\dots,n} \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] |f(b) - f(a)|$$

$$\le \left[\frac{1}{2}\nu(h) + \sup_{i=0,\dots,n} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] |f(b) - f(a)| \le \nu(h)\, |f(b) - f(a)|$$

for all $\xi_i\ (i = 0, ..., n-1)$ as above.

The case of lipschitzian mappings is embodied into the following corollary.

Corollary 3.3. *Let $u : [a,b] \to \mathbf{R}$ be an L-lipschitzian mapping on $[a,b]$ and $I_n, \xi_i\ (i = 0, ..., n-1)$ be as above. Then we have the Riemann quadrature formula* (3.1) *and the remainder satisfies the estimation*

$$|W_n(f, I_n, \boldsymbol{\xi})| \le L \sum_{i=0}^{n-1} \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] h_i$$

$$\le L \sum_{i=0}^{n-1} h_i^2$$

The proof is obvious by Corollary 2.3 applied on the intervals $[x_i, x_{i+1}]$ and summing the obtained inequalities.

We shall omit the details.

Note that the best estimation we can get from (3.2) is that one for which $\xi_i = \frac{x_i + x_{i+1}}{2}$ obtaining the following midpoint formula:

Corollary 3.4. *Let f, I_n be as Theorem 3.1. Then we have the midpoint rule*

$$\int_a^b f(x)dx = M_n(f, I_n) + S_n(f, I_n)$$

where

$$M_n(f, I_n) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i$$

and the remainder $S_n(f, I_n)$ satisfies the estimation

$$|S_n(f, I_n)| \le \frac{1}{2}\nu(h) \bigvee_a^b (f).$$

Similar results can be obtained from Corollaries 3.2 and 3.3.

Remark 3.1. *If we assume that $f : [a,b] \to \mathbf{R}$ is differentiable on (a,b) and whose derivative f' is integrable on (a,b) we can put instead of $\bigvee_a^b(f)$ the L_1-norm $\|f'\|_1$ obtaining the estimation due to Dragomir-Wang from the paper [1].*

4 Applications for Euler's Beta Mapping

Consider the mapping Beta for real numbers

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}dt, \quad p,q>0$$

and the mapping $e_{p,q}(t) := t^{p-1}(1-t)^{q-1}, t \in [0,1]$.

We have for $p,q>1$ that

$$e'_{p,q}(t) = e_{p-1,q-1}(t)[p-1-(p+q-2)t]$$

and as

$$|p-1-(p+q-2)t| \le \max\{p-1,q-1\}$$

for all $t \in [0,1]$, then

$$\left\|e'_{p,q}\right\|_1 \le \max\{p-1,q-1\}\left\|e_{p-2,q-2}\right\|_1 \tag{4.1}$$

$$= \max\{p-1,q-1\}B(p-1,q-1); \qquad p,q>1.$$

The following inequality for Beta mapping holds

Proposition 4.1. *Let $p,q>1$ and $x \in [0,1]$. Then we have the inequality*

$$\left|B(p,q) - x^{p-1}(1-x)^{q-1}\right| \tag{4.2}$$

$$\le \max\{p-1,q-1\}B(p-1,q-1)\left[\frac{1}{2}\left|\,|\,x-\frac{1}{2}\right|\right].$$

The proof follows by Theorem 2.1 applied for the mapping $e_{p,q}$ and taking into account that $\left\|e'_{p,q}\right\|_1$ satisfies the inequality (4.1).

Corollary 4.2. *Let $p,q>1$. Then we have the inequality*

$$\left|B(p,q) - \frac{1}{2^{p+q-2}}\right| \le \frac{1}{2}\max\{p-1,q-1\}B(p-1,q-1).$$

Now, if we apply Theorem 3.1 for the mapping $e_{p,q}$ we get the following approximation of Beta mapping in terms of Riemann sums.

Proposition 4.3. *Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ a sequence of intermediate points for I_n and $p,q>1$. Then we have the formula*

$$B(p,q) = \sum_{i=0}^{n-1} \xi_i^{p-1}(1-\xi_i)^{q-1}h_i + T_n(p,q)$$

where the remainder $T_n(p,q)$ satisfies the estimation

$$|T_n(p,q)|$$

$$\le \max\{p-1,q-1\}\left[\frac{1}{2}\nu(h) + \sup_{i=0,\dots,n}\left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right]B(p-1,q-1)$$

$$\le \max\{p-1,q-1\}\nu(h)B(p-1,q-1).$$

Particularly, if we choose above $\xi_i = \frac{x_i+x_{i+1}}{2}$ $(i = 0, ..., n-1)$ then we get the approximation

$$B(p,q) = \frac{1}{2^{p+q-2}}\sum_{i=0}^{n-1}(x_i+x_{i+1})^{p-1}(2-x_i-x_{i+1})^{q-1} + V_n(p,q)$$

where

$$|V_n(p,q)| \le \frac{1}{2}\max\{p-1,q-1\}\nu(h)B(p-1,q-1).$$

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