COMMENTS ON AN INEQUALITY FOR THE SUM OF POWERS OF POSITIVE INTEGERS

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ABSTRACT. Certain remarks on some inequalities considered by S.S. Dragomir and J. van der Hoek (J. Math. Anal. Appl. **225**(1998), 542-556) are given.

We make certain remarks on some inequalities considered by S.S. Dragomir and J. van der Hoek (J. Math. Anal. Appl. **225**(1998), 542-556).

1. Let $S_p(n) = \sum_{j=1}^{n} j^p$ $(p > 0)$ be the sum of pth powers of the first n positive integers. Put

$$G_p(n) = \frac{S_p(n)}{n^{p+1}} \, (n \geq 1) .$$

Recently, S.S. Dragomir and J. van der Hoek [4] proved the following results:

(i) For $p \geq 1$ one has

$$G_p(n) \geq \frac{(n+1)^p}{\left[(n+1)^{p+1} - n^{p+1}\right]}.$$

(ii) For $p \geq 1$,

$$G_p(n+1) \leq G_p(n) \, (n \geq 1) .$$

(iii) Let $0 \leq a_j \leq 1$ $\left(j = \overline{1,n}\right)$. Then

$$\sum_{j=1}^{n} j^p a_j \geq G_p(n) \left(\sum_{j=1}^{n} a_j\right)^{p+1},$$

for $p \geq 1$ $(p \in \mathbb{R})$.

In fact, (ii) is equivalent with (i), as can be seen by elementary transformations, while (iii) can be deduced from (i) as well. In the afore mentioned paper, the authors obtained interesting applications of (iii) in guessing theory.

2. Inequality (i) is exactly inequality (2) from [9] (with r in place of p), where it is proved that this relation holds true for all $p > 0$, and with strict inequality. This is essentially due to H. Alzer [1]. The history of this inequality is the following: By investigating a problem on Lorentz sequence spaces, in 1988 J.S. Martins [7] discovered certain interesting inequalities for $S_p(n)$. Let

$$L_p(n) = \left[\frac{(n+1)\, S_p(n)}{n S_p(n+1)}\right]^{\frac{1}{p}} \quad (p > 0)$$

and

$$x_n = \frac{(n!)^{\frac{1}{n}}}{((n+1)!)^{\frac{1}{(n+1)}}}; \; y_n = \frac{n}{(n+1)} \quad (n \geq 1) .$$

Martins proved that (for $p > 0$, $n \geq 1$)

$$L_p(n) \leq x_n \tag{0.1}$$

and in 1993, Alzer [1] established the reverse inequality

$$L_p(n) \geq y_n. \tag{0.2}$$

It is not difficult to see that $\lim_{p\to 0} L_p(n) = x_n$, $\lim_{p\to 0} L_p(n) = y_n$ (see for example [5], [6]), so the bounds (0.1) and (0.2) are the best possible. Quite recently the author [9] has obtained a simple method to prove the inequality (0.2) (by showing first that it is equivalent to (i) for $p > 0$). This proof is based on mathematical induction and Cauchy's mean value theorem of differential calculus. We notice that in [4] the method is based on convex functions.

In 1992, G. Bennett [3] proved the inequalities

$$L_p(n) \leq \frac{n+1}{n+2} \text{ for } p \geq 1 \tag{0.3}$$

and

$$L_p(n) \geq \frac{n+1}{n+2} \text{ for } 0 < p \leq 1. \tag{0.4}$$

Since $x_n > \frac{n+1}{n+2}$.(*see* for example [7] or [8]), and $\frac{n+1}{n+2} > \frac{n}{n+1}$, the inequalities (0.3) and (0.4) are refinements of (0.1) and (0.2) for $p \geq 1$ and respectively $0 < p \leq 1$. The proof of (0.3) and (0.4) given in [3] is quite difficult. In [10] we have obtained an easy proof, based on mathematical induction and Lagrange's mean value theorem of differential calculus.

3. An analogous expression introduced in [3] is $Q_p(n)$ with j^{-p} instead of j^p in the definition of $L_p(n)$. It is proved there that

$$Q_p(n) \leq \frac{n+2}{n+1} \ (r > 0,\ n \geq 1). \tag{0.5}$$

A recent refinement of (0.5) has been obtained in [2], namely

$$Q_p(n) \leq \frac{1}{x_n}. \tag{0.6}$$

A new proof of this result, by using the theory of Euler's Gamma function, is due to the author (unpublished). Finally, we note that the inequalities (0.3) − (0.5) have noteworthy applications in the theory of the so-called power means matrices (see [3]).

References

[1] H. ALZER, *On an inequality of H. Minc and L. Sathre,* J. Math. Anal. Appl., **179**(1993), 396-402.

[2] H. ALZER, *Refinement of an inequality of G. Bennett,* Discrete Math., **135**(1994), 39-46.

[3] G. BENNETT, *Lower bounds for matrices,* II, Canad. J. Math., **44**(1992), 54-74.

[4] S.S. DRAGOMIR and J. van der HOEK, *Some new analytic inequalities and their applications in guessing theory,* J. Math. Anal. Appl., **225**(1998), 542-556.

[5] G.H. HARDY, J.E. LITTLEWOOD and G. PÓLYA, *Inequalities,* Cambridge Univ. Press, Cambridge, 1952.

[6] D.S. MITRINOVIĆ, *Analytic Inequalities,* Springer Verlag, 1970.

[7] J.S. MARTINS, *Arithmetic and geometric means, an application to Lorentz sequence spaces,* Math. Nachr., **139**(1988), 281-288.

[8] J. SÁNDOR, *Sur la fonction Gamma,* Center Rech Math. Pures, Neuchâtel, Série I, Fasc. **21**(1989), 4-7.

[9] J. SÁNDOR, *On an inequality of Alzer,* J. Math. Anal. Appl., **192**(1995), 1034-1035.
[10] J. SÁNDOR, *On an inequality of Bennett,* General Mathematics (Sibiu), **3**(1995), No. 3-4, 121-125.

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