HÖLDER INEQUALITY FOR GRAND LEBESGUE SPACES

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ABSTRACT. The Grand L^p space $L^{p)}(\Omega)$ $(1 < p < +\infty)$, $|\Omega| < +\infty$ introduced by Iwaniec-Sbordone is defined as the *Banach Function Space* of the measurable functions f and Ω such that

$$\|f\|_{p)} = \sup_{0<\epsilon<p-1} \left(\varepsilon \frac{1}{|\Omega|} \int_\Omega |f|^{p-\varepsilon}\, dx \right)^{\frac{1}{p-\varepsilon}} < +\infty.$$

In this Note we prove the Hölder-type inequality

$$(*) \qquad \frac{1}{|\Omega|} \int_\Omega fg dx \le \|f\|_{p)} \|g\|_{(p'} \ \ \forall f \in L^{p)}(\Omega),\ \forall g \in L^\infty(\Omega)$$

where

$$\|g\|_{(p'} = \inf_{g=\sum_{k=1}^\infty g_k} \left\{ \sum_{k=1}^\infty \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\frac{1}{|\Omega|} \int_\Omega |g_k|^{(p-\varepsilon)'}\, dx \right)^{\frac{1}{(p-\varepsilon)'}} \right\}$$

The inequality proved is sharp in the following sense: for any f in $L^\infty(\Omega)$ there exists g such that equality holds. It can be proved that the functional $\|\cdot\|_{(p'}$ can be extended into the co-called *dual norm* of the Grand Lebesgue Spaces, therefore, inequality $(*)$ is a special case of the Hölder inequality for Grand Lebesgue Spaces.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \ge 1)$ be a set of Lebesgue measure $|\Omega| < +\infty$ and let $1 < p < +\infty$. The Grand L^p space, that will be denoted by $L^{p)}(\Omega)$, introduced by Iwaniec-Sbordone in [9] is defined as the space of the measurable functions f on Ω such that

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \frac{1}{|\Omega|} \int_\Omega |f|^{p-\varepsilon}\, dx \right)^{\frac{1}{p-\varepsilon}} < +\infty.$$

Notice that for all $1 < p < +\infty$ there exist c_1, $c_2 > 0$ such that

$$c_1 \frac{1}{|\Omega|} \int_\Omega g dx \le \|g\|_{p)} \le c_2 ess \sup_\Omega g.$$

Therefore, the following inclusions hold

Date: May, 1999.
1991 *Mathematics Subject Classification.* 46E30.
Key words and phrases. Grand L^p spaces, small L^p spaces, rearrangement - invariant Banach Function Spaces, dual spaces, decomposition norm, Hölder inequality.

$$L^{\infty}(\Omega) \subset L^{p)}(\Omega) \subset L^{1}(\Omega), \qquad \forall 1 < p < +\infty.$$

Grand L^p spaces have been considered in various fields: in the theory of Partial Differential Equations (see e.g. [10], [11], [12], [13]), in the study of maximal operators and, more generally, quasilinear operators, and in interpolation theory (see e.g. [5], [2]). In particular, in the theory of Partial Differential Equations, it turns out that they are the right spaces in which some nonlinear equations have to be considered (see [8], [6]). Also, they have been studied in their own, various properties have been developed e.g. in [7], [3].

It is easy to construct some functionals $\mathcal{N}_{p'}$ such that a sort of Hölder inequality for Grand L^p spaces holds. For instance, setting

$$\mathcal{N}_{p'}(g) = \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\frac{1}{|\Omega|} \int_{\Omega} |g|^{(p-\varepsilon)'} dx \right)^{\frac{1}{(p-\varepsilon)'}},$$

we have easily (see the proof of Theorem 1)

$$\frac{1}{|\Omega|} \int_{\Omega} fg dx \leq \|f\|_{p)} \mathcal{N}_{p'}(g). \tag{1.1}$$

Unfortunately, $\mathcal{N}_{p'}$ is not a norm, because the triangular inequality fails. The "best" functional $\mathcal{N}_{p'}$ such that (1.1) holds is the so-called "dual norm" (see [1]) of that one defining the Grand L^p spaces, i.e.,

$$g \to \sup_{\substack{g \not\equiv 0 \\ f \in L^{p)}(\Omega)}} \mathcal{N}_{p'}(g) \frac{\frac{1}{|\Omega|} \int_{\Omega} fg dx}{\|f\|_{p)}}.$$

We found an expression of the dual norm free from the definition of the norm in Grand L^p spaces in [4]. In the case $g \in L^{\infty}(\Omega)$, this norm is given by

$$\|g\|_{(p'} = \inf_{g = \sum_{k=1}^{\infty} g_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\frac{1}{|\Omega|} \int_{\Omega} |g_k|^{(p-\varepsilon)'} dx \right)^{\frac{1}{(p-\varepsilon)'}} \right\}.$$

In this Note, we prove that the following Hölder-type inequality holds

$$\frac{1}{|\Omega|} \int_{\Omega} fg dx \leq \|f\|_{p)} \|g\|_{(p'} \ \forall f \in L^{p)}(\Omega), \ \forall g \in L^{\infty}(\Omega), \tag{1.2}$$

and that such inequality is sharp in the sense that for any f in $L^{\infty}(\Omega)$ there exists g such that equality holds. Notice that inequality (1.2) is true even for g belonging to a bigger space, which is a Banach Function Space related to the dual space of the Grand Lebesgue Space (see [4]).

In order to have a simpler notation, unless otherwise specified, all the spaces considered in the sequel have to be assumed as spaces of functions on Ω. Therefore for instance, we will write $L^{p)}$ instead of $L^{p)}(\Omega)$, L^{∞} instead of $L^{\infty}(\Omega)$, etc.

Theorem 1. *The following Hölder-type inequality holds:*

$$\frac{1}{|\Omega|}\int_\Omega fg dx \le \|f\|_{p)}\|g\|_{(p'}\ \ \forall f\in L^{p)},\ \forall g\in L^\infty .$$

Proof. Let $|g|=\sum_{k=1}^\infty g_k$ for any decomposition with $g_k\ge 0\ \forall k\in\mathbb{N}$ and let $f\in L^p$.

For each $k\in\mathbb{N}$ and for each $0<\varepsilon<p-1$ we have

$$\begin{aligned}\frac{1}{|\Omega|}\int_\Omega fg_k dx &\le \left(\frac{1}{|\Omega|}\int_\Omega |f|^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}}\left(\frac{1}{|\Omega|}\int_\Omega |g_k|^{(p-\varepsilon)'}dx\right)^{\frac{1}{(p-\varepsilon)'}}\\ &= \left(\varepsilon\cdot\frac{1}{|\Omega|}\int_\Omega |f|^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}}\cdot\varepsilon^{-\frac{1}{p-\varepsilon}}\left(\frac{1}{|\Omega|}\int_\Omega |g_k|^{(p-\varepsilon)'}dx\right)^{\frac{1}{(p-\varepsilon)'}}\\ &\le \varepsilon^{-\frac{1}{p-\varepsilon}}\left(\frac{1}{|\Omega|}\int_\Omega |g_k|^{(p-\varepsilon)'}dx\right)^{\frac{1}{(p-\varepsilon)'}}\|f\|_{p)}\end{aligned}$$

and therefore

$$\frac{1}{|\Omega|}\int_\Omega fg_k dx \le \inf_{0<\varepsilon<p-1}\varepsilon^{-\frac{1}{p-\varepsilon}}\left(\frac{1}{|\Omega|}\int_\Omega |g_k|^{(p-\varepsilon)'}dx\right)^{\frac{1}{(p-\varepsilon)'}}\|f\|_{p)},$$

from which

$$\begin{aligned}\frac{1}{|\Omega|}\int_\Omega fg dx &\le \frac{1}{|\Omega|}\int_\Omega |f|\left|\sum_{k=1}^\infty g_k\right|dx \le \sum_{k=1}^\infty \frac{1}{|\Omega|}\int_\Omega |f|\,g_k dx\\ &\le \sum_{k=1}^\infty \inf_{0<\varepsilon<p-1}\varepsilon^{-\frac{1}{p-\varepsilon}}\left(\frac{1}{|\Omega|}\int_\Omega |g_k|^{(p-\varepsilon)'}dx\right)^{\frac{1}{(p-\varepsilon)'}}\|f\|_{p)}.\end{aligned}$$

Hence

$$\frac{1}{|\Omega|}\int_\Omega fg dx \le \|g\|_{(p'}\|f\|_{p)}.$$

Theorem 1 is therefore proved.

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Theorem 2. *Let f be any function in L^∞. Then there exists $g\in L^\infty$ such that*

$$\frac{1}{|\Omega|}\int_\Omega fg dx = \|f\|_{p)}\|g\|_{(p'}.$$

Proof. If $f\in L^\infty$, then

$$\lim_{\varepsilon\to 0}\left(\varepsilon\frac{1}{|\Omega|}\int_\Omega |f|^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}} = 0$$

therefore

$$\sup_{0<\varepsilon<p-1}\left(\varepsilon\frac{1}{|\Omega|}\int_{\Omega}|f|^{p-\varepsilon}\,dx\right)^{\frac{1}{p-\varepsilon}}=\left(\sigma\frac{1}{|\Omega|}\int_{\Omega}|f|^{p-\sigma}\,dx\right)^{\frac{1}{p-\sigma}}=\|f\|_{p)}$$

where $\sigma=\sigma(f)\in\,]0,p-1[\,.$

Let $g\in L^{\infty}$ be such that

$$\frac{1}{|\Omega|}\int_{\Omega}fgdx=\left(\frac{1}{|\Omega|}\int_{\Omega}|f|^{p-\sigma}\,dx\right)^{\frac{1}{p-\sigma}}\left(\frac{1}{|\Omega|}\int_{\Omega}|g|^{(p-\sigma)'}\,dx\right)^{\frac{1}{(p-\sigma)'}}$$

We have

$$\begin{aligned}
&\frac{1}{|\Omega|}\int_{\Omega}fgdx\\
\leq\;&\|f\|_{p)}\,\|g\|_{(p'}\\
=\;&\|f\|_{p)}\inf_{g=\sum_{k=1}^{\infty}g_k}\left\{\sum_{k=1}^{\infty}\inf_{0<\varepsilon<p-1}\varepsilon^{-\frac{1}{p-\varepsilon}}\left(\frac{1}{|\Omega|}\int_{\Omega}|g_k|^{(p-\varepsilon)'}\,dx\right)^{\frac{1}{(p-\varepsilon)'}}\right\}\\
\leq\;&\|f\|_{p)}\inf_{0<\varepsilon<p-1}\varepsilon^{-\frac{1}{p-\varepsilon}}\left(\frac{1}{|\Omega|}\int_{\Omega}|g|^{(p-\varepsilon)'}\,dx\right)^{\frac{1}{(p-\varepsilon)'}}\\
\leq\;&\|f\|_{p)}\,\sigma^{-\frac{1}{p-\sigma}}\left(\frac{1}{|\Omega|}\int_{\Omega}|g|^{(p-\sigma)'}\,dx\right)^{\frac{1}{(p-\sigma)'}}\\
=\;&\sigma^{\frac{1}{p-\sigma}}\cdot\left(\frac{1}{|\Omega|}\int_{\Omega}|f|^{p-\sigma}\,dx\right)^{\frac{1}{p-\sigma}}\cdot\sigma^{-\frac{1}{p-\sigma}}\left(\frac{1}{|\Omega|}\int_{\Omega}|g|^{(p-\sigma)'}\,dx\right)^{\frac{1}{(p-\sigma)'}}\\
=\;&\frac{1}{|\Omega|}\int_{\Omega}fgdx.
\end{aligned}$$

Therefore the inequalities are in fact equalities, the first inequality is equality and the theorem is proved.

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