# ON CERTAIN INEQUALITIES FOR MEANS, III

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ABSTRACT. A sequential method is applied to obtain inequalities between a mean introduced by H.-J. Seiffert [9], and other means, including the logarithmic mean, the identric mean and the arithmetic-geometric mean of Gauss.

## 1. INTRODUCTION

Let $x,\ y$ be positive real numbers. The logarithmic mean and the identric mean of $x$ and $y$ are defined by

$$L = L(x,y) = \frac{x-y}{\log x - \log y} \text{ for } x \neq y;\ L(x,x) = x, \tag{1.1}$$

and

$$I = I(x,y) = \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}} \text{ for } x \neq y;\ I(x,x) = x, \tag{1.2}$$

respectively. Let $A = A(x,y) = \frac{x+y}{2}$ and $G = G(x,y) = \sqrt{xy}$ denote the arithmetic, resp. geometric mean of $x$ and $y$.

It is well-known that for $x \neq y$, one has (see e.g. [4])

$$G < L < I < A. \tag{1.3}$$

In 1993, H.-J. Seiffert [9] introduced the mean

$$P = P(x,y) = \frac{x-y}{4\arctan\left(\sqrt{\frac{x}{y}}\right) - \pi} \text{ for } x \neq y;\ P(x,x) = x.$$

Seiffert [9] proved that for $x \neq y$

$$L < P < I \tag{1.4}$$

and later [10], by using certain series representations:

$$\frac{1}{P} < \frac{1}{3}\left(\frac{1}{G} + \frac{2}{A}\right), \tag{1.5}$$

$$GA < LP, \tag{1.6}$$

$$P < A < \frac{\pi}{2}P. \tag{1.7}$$

In fact, $P$ can be also be written in the equivalent form

$$P(x,y) = \frac{x-y}{2\arcsin\frac{x-y}{x+y}} \text{ for } x \neq y \tag{1.8}$$

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(see [8]). Clearly, we may suppose $0 < x < y$, and we note that (1.8) implies

$$\frac{A}{P} = \frac{\arcsin z}{z} = f(z),$$

where $z = \frac{x-y}{x+y}$, $0 < z < 1$; and $f$ being a strictly increasing function, clearly

$$1 = \lim_{x \to 0} f(z) < \frac{A}{P} < \lim_{z \to 1} f(z) = \frac{\pi}{2},$$

giving $(1.7)$.

Another remark is that (1.8) can be written also as

$$P(x,y) = \frac{2}{x-y} \arccos\left(\frac{2}{x-y}\sqrt{xy}\right) = \frac{2}{x-y} \arccos \frac{x_0}{y_0} \tag{1.9}$$

where $x_0 = \sqrt{xy}$, $y_0 = \frac{x+y}{2}$. Since $x_0 < y_0$, $P$ is the common limit of a pair of sequences given by

$$x_{n+1} = \frac{x_n + y_n}{2},\ y_{n+1} = \sqrt{x_{n+1} y_n},\ n = 0, 1, \ldots \tag{1.10}$$

(see [1, p. 498]). According to B. C. Carlson [1], the algorithm (1.10) is due to Pfaff (see also [3]), who determined the common limit (1.9) of the sequences $(x_n)$ and $(y_n)$.

By using the sequential method from part II of this series (see [6], [7]), we will be able, in what follows, to improve relations $(1.4)-(1.6)$, and obtain other inequalities related to the mean $P$.

## 2. Gauss', Borchardt's and Pfaff's Algorithms

Pfaff's algorithm is given by $(1.10)$, where $x_0 = \sqrt{xy}$, $y_0 = \frac{x+y}{2}$. Let us denote the Borchardt algorithm by

$$a_{n+1} = \frac{a_n + b_n}{2},\ b_{n+1} = \sqrt{a_{n+1} b_n}\ (n \geq 1),\ a_0 = x,\ b_0 = y,\ b_1 = \sqrt{xy} \tag{2.1}$$

and the Gauss algorithm by

$$f_{n+1} = \frac{f_n + g_n}{2},\ g_{n+1} = \sqrt{g_n f_n}\ (n \geq 0),\ f_0 = x,\ g_0 = y. \tag{2.2}$$

It is well known that $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = L(x,y)$ - the logarithmic mean of $x$ and $y$; $\lim_{n \to \infty} f_n = \lim_{n \to \infty} g_n = M(x,y)$ - the famous arithmetic-geometric mean of Gauss (see e.g. [1], [2], [3], [7]). M. K. Vamanamurthy and M. Vourinen ( [11]) have proved that

$$L < M < \frac{\pi}{2} L \tag{2.3}$$

$$M < I < A \tag{2.4}$$

$$M < \frac{A+G}{2} \tag{2.5}$$

and the author [7] has obtained refinements, based on the Gauss and Borchardt algorithm.

The aim of this paper is to offer new proof of $(1.4)$, $(1.5)$, $(1.6)$, and, in fact, to obtain strong refinements of these relations.

## 3. Monotonicity Properties and Applications

**Theorem 1.** *For all $n \geq 0$ we have*

$$x_n < P < y_n. \tag{3.1}$$

*Particularly,*

$$\frac{A+G}{2} < P < \sqrt{\left(\frac{A+G}{2}\right)A}. \tag{3.2}$$

*Proof.* Since $y_0 > x_0$ and $y_{n+1} > x_{n+1}$ iff $\frac{x_n+y_n}{2} > \sqrt{\frac{x_n+y_n}{2} y_n}$, i.e., $y_n > x_n$, by induction it follows that $y_n > x_n$ for all $n$. The inequality $x_{n+1} > x_n$ is equivalent to $y_n > x_n$, while $y_{n+1} < y_n$ to $x_{n+1} < y_n$, i.e., $\frac{x_n+y_n}{2} < y_n$, thus $x_n < y_n$, which is proved. We have proved that the sequence $(x_n)_{n\geq 0}$ is strictly increasing, $(y_n)_{n\geq 0}$ is strictly decreasing, having the same limit $P$, so (3.1) follows. From $x_1 = \frac{A+G}{2}$, $y_1 = \sqrt{x_1 y_0} = \sqrt{\frac{A+G}{2} A}$ we obtain relation (3.2). ∎

**Corollary 1.**

$$L < M < \frac{A+G}{2} < P. \tag{3.3}$$

*This follows by* (2.3), (2.5) *and* (3.2).

**Remark 1.** *Relation $L < \frac{A+G}{2}$ follows also from the known fact (see e.g.* [4], [6]*) that $L < A_{\frac{1}{3}} < A_{\frac{1}{2}} = \frac{A+G}{2}$, where $A_s = A_s(x,y) = \left(\frac{x^s+y^s}{2}\right)^{\frac{1}{s}}$.*

**Theorem 2.** *For all $n \geq 0$, we have*

$$\sqrt[3]{y_n^2 x_n} < P < \frac{x_n + 2y_n}{3}. \tag{3.4}$$

*Particularly,*

$$\sqrt[3]{A^2 G} < P < \frac{G+2A}{3}. \tag{3.5}$$

*Proof.* One has $y_{n+1}^2 x_{n+1} = (x_{n+1} y_n) x_{n+1} = x_{n+1}^2 y_n > y_n^2 x_n$ iff $x_{n+1}^2 > x_n y_n$ i.e. $\left(\frac{x_n+y_n}{2}\right)^2 > x_n y_n$, which is true, Thus, the sequence $\left(y_n^2 x_n\right)_{n\geq 0}$ is strictly increasing, having as limit $P^3$. This gives the first part of (3.4). Next, from $x_{n+1} + 2y_{n+1} = \frac{x_n+y_n}{2} + 2\sqrt{\frac{x_n+y_n}{2} y_n} < \frac{x_n+y_n}{2} + \frac{x_n+y_n}{2} + y_n$ (by $2\sqrt{uv} < u+v$ for $u \neq v$) we get that the sequence $(x_n + 2y_n)_{n\geq 0}$ is strictly decreasing, having the limit $3P$. This implies the second part of (3.4). For $n = 0$, we obtain the double inequality (3.5). ∎

**Corollary 2.**

$$\frac{AG}{L} < \sqrt[3]{A^2 G} < P < \frac{G+2A}{3} < I. \tag{3.6}$$

*The first inequality is a consequence of $L > \sqrt[3]{G^2 A}$, due to Leach and Sholander (for refinements, see* [7]*), and the last inequality is due to the author (see* [6]*).*

**Remark 2.** *The left side of* $(3.6)$ *improves inequality* $(1.6)$. *Similarly, the left side of* $(3.5)$ *improves inequality* $(1.5)$. *Indeed,*

$$\frac{1}{3}\left(\frac{1}{G}+\frac{2}{A}\right)=\frac{1}{3}\left(\frac{1}{G}+\frac{1}{A}+\frac{1}{A}\right)>\sqrt[3]{\frac{1}{G}\cdot\frac{1}{A^2}}>\frac{1}{P}$$

*by the arithmetic-geometric inequality* $\frac{x+y+z}{3}>\sqrt[3]{xyz}$.

**Remark 3.** *A better estimate for the right side of* $(3.6)$ *can be obtained by applying* $(3.4)$, *e.g., for* $n=1$. *Since* $x_1=\frac{A+G}{2}$, $y_1=\sqrt{\frac{A+G}{2}A}$, *we obtain*

$$P<\frac{1}{3}\left(\frac{A+G}{2}+2\sqrt{\frac{A+G}{2}A}\right)<\frac{G+2A}{3}<I. \tag{3.7}$$

*In an analogous way, the left side of* $(3.4)$ *gives*

$$\left[\left(\frac{A+G}{2}\right)^2A\right]^{\frac{1}{3}}<P. \tag{3.8}$$

*This follows by the remark that*

$$\sqrt[3]{y_1^2x_1}=\sqrt[3]{\left(\frac{A+G}{2}\right)^2A}.$$

**Theorem 3.** *One has*

*a.*

$$\begin{aligned} P\left(x^k,y^k\right) &\geq (P(x,y))^k \text{ and} \\ M\left(x^k,y^k\right) &\geq (M(x,y))^k \text{ for all } k\geq 1. \end{aligned} \tag{3.9}$$

*b.*

$$P\left(x^k,y^k\right)>A_{\frac{k}{2}}^k>A^k>(P(x,y))^k \quad \text{for all } k\geq 2. \tag{3.10}$$

*c.*

$$M\left(x^k,y^k\right)>\frac{1}{k}\frac{x^k-y^k}{x-y}L(x,y)\geq A^{k-1}L>(M(x,y))^k \quad \text{for all } k\geq 2. \tag{3.11}$$

*Proof.* 1.

*a.* As in [7] (for the mean $M$, with $k=2$) we consider the sequences $(p_n)$, $(q_n)$ defined by

$$p_0=\sqrt{x^ky^k},\ q_0=\frac{x^k+y^k}{2},\ p_{n+1}=\frac{p_n+q_n}{2},\ q_{n+1}=\sqrt{p_{n+1}q_n}.$$

Clearly, $\lim_{n\to\infty}p_n=\lim_{n\to\infty}q_n=P\left(x^k,y^k\right)$.
We prove inductively that

$$p_n\geq x_n^k,\ q_n\geq y_n^k \text{ for all } n\geq 0,\ k\geq 1.$$

We have $p_0=x_0^k$ and $q_0\geq y_0^k$ since $\frac{x^k+y^k}{2}\geq\left(\frac{x+y}{2}\right)^k$, which follows by the convexity of the function $t\mapsto t^k$ $(k\geq 1)$. Then, if (3.12) is valid for an $n$, we can write

$$p_{n+1}=\frac{p_n+q_n}{2}\geq\frac{x_n^k+y_n^k}{2}\geq\left(\frac{x_n+y_n}{2}\right)^k=x_{n+1}^k,$$

and

$$q_{n+1} = \sqrt{p_{n+1}q_n} \geq \sqrt{x_{n+1}^k y_n^k} = y_{n+1}^k,$$

i.e. (3.12) is valid for $n+1$, too. By taking $n \to \infty$ in (3.12), we get the first part of $(a)$. The second part can be proved in a completely analogous way.

*b.* By writing the right side of inequality (3.3) for $x^k,\ y^k$ in place of $x,\ y$, one has

$$P\left(x^k, y^k\right) > \left(\frac{\sqrt{x^k}+\sqrt{y^k}}{2}\right)^2 = \left[\left(\frac{x^{\frac{k}{2}}+y^{\frac{k}{2}}}{2}\right)^{\frac{2}{k}}\right]^2 = A_{\frac{k}{2}}^k \geq A_1^k = A^k$$

for $k \geq 2$ (since $A_s$ in increasing in $s$), by $A > P$, we get $(b)$.

*c.* Finally, for $(c)$ remark that $L\left(x^k, y^k\right) < M\left(x^k, y^k\right)$, but

$$L\left(x^k, y^k\right) = \frac{x^k - y^k}{\ln x^k - \ln y^k} = L\frac{x^k - y^k}{k\left(x-y\right)}.$$

We shall prove that

$$(M\left(x,y\right))^k < \frac{x^k - y^k}{k\left(x-y\right)}L\left(x,y\right) \text{ for } k \geq 2. \tag{$*$}$$

First, we note that the function $t \mapsto t^{k-1}$ is convex for $k \geq 2$, so by Hadamard's inequality

$$\int_y^x f\left(t\right)dt \geq \left(x-y\right)f\left(\frac{x+y}{2}\right) \ (y < x)$$

we get

$$\frac{x^k - y^k}{k\left(x-y\right)} \geq A^{k-1}.$$

It is sufficient to prove that

$$M < A^{k-1}L\left(\leq \frac{x^k - y^k}{k\left(x-y\right)}L\right).$$

It is known that (see [11], [7]) $M^2 < AL$, i.e. $M^k < A^{\frac{k}{2}}L^{\frac{k}{2}} \leq A^{k-1}L$, since this is equivalent to $L^{\frac{k}{2}-1} \leq A^{\frac{k}{2}-1}$, valid by $L < A$ and $k \geq 2$. So $(*)$ holds true, and this finishes the proof of (3.11).

∎

Finally, we prove

**Theorem 4.** 1.

*a) For all* $k > 1$ *we have*

$$P\left(x^k, y^k\right) < \frac{x^k + y^k}{x+y}P\left(x,y\right). \tag{3.12}$$

*b) For* $0 < k < 2$ *we have*

$$P\left(x^k, y^k\right) > \frac{x^k + y^k}{x^2+y^2}P\left(x^2,y^2\right). \tag{3.13}$$

*c) For all* $k > 0$,

$$P\left(x^{k+1}, y^{k+1}\right) < \frac{x^{k+1} + y^{k+1}}{x^k + y^k} P\left(x^k, y^k\right). \tag{3.14}$$

*Proof.* We have seen in the Introduction that

$$\frac{A}{P} = \frac{\arcsin z}{z} = f(z), \text{ where } z = \frac{x-y}{x+y} \ (0 < y < x)$$

is an increasing function of $z$. Since $\frac{x^k-y^k}{x^k+y^k} > \frac{x-y}{x+y}$ for $k > 1$ (and $0 < y < x$), we get $\frac{A(x^k,y^k)}{P(x^k,y^k)} > \frac{A(x,y)}{P(x,y)}$, giving (3.12).

Relation (3.13) follows from $\frac{x^k-y^k}{x^k+y^k} < \frac{x^2-y^2}{x^2+y^2}$ for $k < 2$ in the same manner. Finally, (3.14) is a consequence of $\frac{x^{k+1}-y^{k+1}}{x^{k+1}+y^{k+1}} > \frac{x^k-y^k}{x^k+y^k}$. ∎

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