NEW INEQUALITIES FOR CONVEX FUNCTIONS WITH APPLICATIONS FOR THE N-ENTROPY OF A DISCRETE RANDOM VARIABLE

S. S. DRAGOMIR

Abstract. New inequalitieis for convex mappings of a real variable and applications in Information Theory for Shannon's entropy are given.

1. Introduction

The following converse of Jensen's discrete inequality for convex mappings of a real variable has been proved in 1994 by S.S. Dragomir and N.M. Ionescu [?]:

Theorem 1. *Let $f : I \subseteq R \to R$ be a convex function on the interval I, $x_i \in \mathring{I}$ ($\mathring{I}$ is the interior of I), $p_i \geq 0$, $(i = 1, ..., n)$ and $\sum_{i=1}^{n} p_i = 1$. Then we have the inequality*

$$
\begin{aligned}
(1.1) \qquad 0 &\leq \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \\
&\leq \sum_{i=1}^{n} p_i x_i f_{+}^{'}(x_i) - \sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i f_{+}^{'}(x_i)
\end{aligned}
$$

where $f_{+}^{'}$ is the right derivative of f on $\mathring{I}$.

They also pointed out some natural application of (**??**) in connection to the arithmetic mean - geometric mean inequality, to the generalized polygonal inequality etc.

A generalization of (**??**) for differentiable convex mappings of several variables has been obtained in 1996 by S.S. Dragomir and C.J. Goh in [?].

Theorem 2. *Let $f : K \subseteq R^m \to R$ be a differentiable convex mapping on the convex set K, $\bar{x}_i \in K$ $(i = 1, ..., n)$, $p_i \geq 0$ $(i = 1, ..., n)$ with $\sum_{i=1}^{n} p_i = 1$. Then*

$$
\begin{aligned}
(1.2) \qquad 0 &\leq \sum_{i=1}^{n} p_i f(\bar{x}_i) - f\left(\sum_{i=1}^{n} p_i \bar{x}_i\right) \\
&\leq \sum_{i=1}^{n} p_i \langle \nabla f(\bar{x}_i), \bar{x}_i \rangle - \left\langle \sum_{i=1}^{n} p_i \nabla f(\bar{x}_i), \sum_{i=1}^{n} p_i \bar{x}_i \right\rangle
\end{aligned}
$$

where $\langle ., . \rangle$ is the usual inner product on R^m and $\nabla f(\bar{x}) = \left(\frac{\delta f(\bar{x})}{\delta x_1}, ..., \frac{\delta f(\bar{x})}{\delta x_m}\right)$.

Date: 17 March, 1999.
1991 *Mathematics Subject Classification.* Primary 26 D 15; Secondary 94 X xx.
Key words and phrases. Convex Functions, Jensen's Inequality, Entropy Mapping.

The authors applied the inequality (**??**) in Information Theory for the entropy mapping, conditional entropy, mutual information etc.

An integral version of (**??**) has been employed by S.S. Dragomir and C.J. Goh [**?**] to obtain different bounds for the entropy, conditional entropy and mutual information of continuous random variable. Also, some applications in torsion theory of a similar result which counterparts Hadamard's inequality, has been given by S.S. Dragomir and G. Keady in 1996 [**?**].

For recent generalizations, both for the discrete case and continuous case as well as extensions for mappings defined on normed linear spaces, we recommend M. Matić's Ph.D. Dissertation [**?**], where further applications in Information Theory have been given.

In paper [**?**], the authors obtained the following upper bound for the first member in (**??**),

Theorem 3. *Let* $f : K \subseteq R^m \to R$ *be a differentiable convex mapping on the convex set* K, $\bar{x}_i \in K$ $(i = 1, ..., n)$, $p_i \geq 0$, $(i = 1, ..., n)$ *with* $\sum_{i=1}^{n} p_i = 1$. *Then*

(1.3)
$$\begin{aligned}
0 &\leq \sum_{i=1}^{n} p_i f(\bar{x}_i) - f\left(\sum_{i=1}^{n} p_i \bar{x}_i\right) \\
&\leq \begin{cases}
\max\limits_{1\leq i<j\leq n} \|\bar{x}_i - \bar{x}_j\| \sum\limits_{1\leq i<j\leq n} p_i p_j \|\nabla f(\bar{x}_i) - \nabla f(\bar{x}_j)\| ; \\
\left(\sum\limits_{1\leq i<j\leq n} p_i p_j \|\bar{x}_i - \bar{x}_j\|^p\right)^{1/p} \left(\sum\limits_{1\leq i<j\leq n} p_i p_j \|\nabla f(\bar{x}_i) - \nabla f(\bar{x}_j)\|^q\right)^{1/q}, \\
\qquad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\
\max\limits_{1\leq i<j\leq n} \|\nabla f(\bar{x}_i) - \nabla f(\bar{x}_j)\| \sum\limits_{1\leq i<j\leq n} p_i p_j \|\bar{x}_i - \bar{x}_j\|
\end{cases} \\
&\leq \frac{1}{2} \max_{1\leq i<j\leq n} \|\bar{x}_i - \bar{x}_j\| \max_{1\leq i<j\leq n} \|\nabla f(\bar{x}_i) - \nabla f(\bar{x}_j)\|
\end{aligned}$$

where $\|\cdot\|$ *is the usual Euclidean norm on* R^m.

and applied it for the entropy mapping in Information Theory.

In [**?**], the authors pointed out other converse inequalities which are more useful in applications.

Theorem 4. *Let $f : K \subseteq R^m \to R$ be a differentiable convex mapping on the convex set K, $\bar{x}_i \in K$ $(i = 1, ..., n)$, $p_i \geq 0$, $(i = 1, ..., n)$ with $\sum_{i=1}^{n} p_i = 1$. Then*

(1.4)

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} p_i f(\bar{x}_i) - f\left(\sum_{i=1}^{n} p_i \bar{x}_i\right) \\
&\leq \begin{cases}
\max\limits_{i=1,..,n} \left\| \bar{x}_i - \sum\limits_{j=1}^{n} p_j \bar{x}_j \right\| \sum\limits_{i=1}^{n} p_i \left\| \nabla f(\bar{x}_i) - \sum\limits_{j=1}^{n} p_j \nabla f(\bar{x}_j) \right\|; \\
\left(\sum\limits_{i=1}^{n} p_i \left\| \bar{x}_i - \sum\limits_{j=1}^{n} p_j \bar{x}_j \right\|^p \right)^{1/p} \left(\sum\limits_{i=1}^{n} p_i \left\| \nabla f(\bar{x}_i) - \sum\limits_{j=1}^{n} p_j \nabla f(\bar{x}_j) \right\|^q \right)^{1/q}, \\
\qquad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\
\max\limits_{i=1,..,n} \left\| \nabla f(\bar{x}_i) - \sum\limits_{j=1}^{n} p_j \nabla f(\bar{x}_j) \right\| \sum\limits_{i=1}^{n} p_i \left\| \bar{x}_i - \sum\limits_{j=1}^{n} p_j \bar{x}_j \right\|
\end{cases} \\
&\leq \max_{i=1,..,n} \left\| \bar{x}_i - \sum_{i=1}^{n} p_j \nabla f(\bar{x}_j) \right\| \max_{i=1,..,n} \left\| \sum_{i=1}^{n} \nabla f(\bar{x}_i) - \sum_{j=1}^{n} p_j \nabla f(\bar{x}_j) \right\|.
\end{aligned}
$$

In this paper, we point out other general inequalities for convex mappings of a real variable and apply them in Information Theory for the entropy mapping.

2. Some Results for Convex Functions of a Real Variable

We shall define the following sequence of functionals:

$$\mathfrak{I}_n : Conv\,(I) \times S_+^*\,(\mathbb{R}) \times \mathfrak{P}_f^*\,(\mathbb{N}) \times S\left(\mathring{\mathrm{I}}\right) \longrightarrow \mathbb{R}$$

given by:

$$\mathfrak{I}_n\,(f, p, H, x) := \frac{1}{P_H^n} \sum_{i_1,...,i_n \in H} p_{i_1}...p_{i_n} f\left(\frac{x_{i_1} + ... + x_{i_n}}{n}\right),$$

where

$$Conv\,(I) := \{f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}\,|f \text{ is convex on the interval } I\};$$

$$S_+^*\,(\mathbb{R}) := \left\{p = (p_i)_{i\in\mathbb{N}}\,|p_i > 0 \text{ for all } i \in \mathbb{N}\right\};$$

$$\mathfrak{P}_f^*\,(\mathbb{N}) := \{H \subset \mathbb{N}\,|H \text{ is finite and } H \neq \emptyset\};$$

$$S\left(\mathring{\mathrm{I}}\right) := \left\{x = (x_i)_{i\in\mathbb{N}} \middle| x_i \in \mathring{\mathrm{I}} \text{ for all } i \in \mathbb{N}, \mathring{\mathrm{I}} \text{ is the interior of } I\right\}.$$

The following estimation holds:

Theorem 5. *Let $f'_+ : \mathring{I} \to \mathbb{R}$ be the right derivative of the convex function f. Define the sequence of mappings:*

$$
\begin{aligned}
E_n\,(f, p, H, x) \quad : \quad &= \frac{1}{P_H^n} \sum_{i_1,...,i_n \in H} p_{i_1}...p_{i_n} f'_+\left(\frac{x_{i_1} + ... + x_{i_n}}{n}\right) x_{i_1} \\
&\quad - \frac{1}{P_H} \sum_{i\in H} p_i x_i \frac{1}{P_H^n} \sum_{i_1,...,i_n \in H} p_{i_1}...p_{i_n} f'_+\left(\frac{x_{i_1} + ... + x_{i_n}}{n}\right)
\end{aligned}
$$

for $(f,p,H,x) \in Conv(I) \times S_{+}^{*}(\mathbb{R}) \times \mathfrak{P}_{f}^{*}(\mathbb{N}) \times S\left(\overset{\circ}{I}\right)$. *Then we have the inequalities:*

$$0 \le \mathfrak{I}_n(f,p,H,x) - \mathfrak{I}_{n+1}(f,p,H,x) \le \frac{1}{n+1} E_n(f,p,H,x) \tag{2.1}$$

and

$$0 \le \mathfrak{I}_n(f,p,H,x) - f\left(\frac{1}{P_H}\sum_{i\in H} p_i x_i\right) \le E_n(f,p,H,x) \tag{2.2}$$

for all $n \ge 1$ *and* $(f,p,H,x) \in Conv(I) \times S_{+}^{*}(\mathbb{R}) \times \mathfrak{P}_{f}^{*}(\mathbb{N}) \times S\left(\overset{\circ}{I}\right)$.

Proof. By the convexity of f on I, we have the inequality:

$$f'_{+}(y)(x-y) \le f(x) - f(y) \le f'_{+}(x)(x-y) \tag{2.3}$$

for all $x, y \in \mathring{\mathrm{I}}$.

If we choose in (**??**)

$$x = \frac{x_{i_1} + ... + x_{i_n}}{n} \in \mathring{\mathrm{I}} \text{ and } y = \frac{x_{i_1} + ... + x_{i_{n+1}}}{n+1} \in \mathring{\mathrm{I}},$$

where $i_1, ..., i_{n+1} \in H$, we deduce:

$$\begin{aligned} & f'_{+}\left(\frac{x_{i_1} + ... + x_{i_{n+1}}}{n+1}\right)\left[\frac{x_{i_1} + ... + x_{i_n} - n x_{i_{n+1}}}{n(n+1)}\right] \\ \le\ & f\left(\frac{x_{i_1} + ... + x_{i_n}}{n}\right) - f\left(\frac{x_{i_1} + ... + x_{i_{n+1}}}{n+1}\right) \\ \le\ & f'_{+}\left(\frac{x_{i_1} + ... + x_{i_n}}{n}\right)\left[\frac{x_{i_1} + ... + x_{i_n} - n x_{i_{n+1}}}{n(n+1)}\right] \end{aligned} \tag{2.4}$$

for all $i_1, ..., i_{n+1} \in H$.

If we multiply (**??**) by $p_{i_1}...p_{i_{n+1}} \ge 0$ and summing over $i_1, ..., i_{n+1} \in H$ we deduce:

$$\begin{aligned} A \quad : & = \frac{1}{P_H^{n+1}} \sum_{i_1,...,i_{n+1}\in H} p_{i_1}...p_{i_{n+1}} f'_{+}\left(\frac{x_{i_1} + ... + x_{i_{n+1}}}{n+1}\right) \\ & \times \left[\frac{x_{i_1} + ... + x_{i_n} - n x_{i_{n+1}}}{n(n+1)}\right] \\ & \le \mathfrak{I}_n(f,p,H,x) - \mathfrak{I}_{n+1}(f,p,H,x) \\ & \le \frac{1}{P_H^{n+1}} \sum_{i_1,...,i_{n+1}\in H} p_{i_1}...p_{i_{n+1}} f'_{+}\left(\frac{x_{i_1} + ... + x_{i_n}}{n}\right) \\ & \times \left[\frac{x_{i_1} + ... + x_{i_n} - n x_{i_{n+1}}}{n(n+1)}\right] \\ : & = B. \end{aligned} \tag{2.5}$$

As, it is clear that

$$\begin{aligned} & \frac{1}{P_H^{n+1}} \sum_{i_1,...,i_{n+1}\in H} p_{i_1}...p_{i_{n+1}} f'_{+}\left(\frac{x_{i_1} + ... + x_{i_{n+1}}}{n+1}\right) x_{i_k} \\ = & \frac{1}{P_H^{n+1}} \sum_{i_1,...,i_{n+1}\in H} p_{i_1}...p_{i_{n+1}} f'_{+}\left(\frac{x_{i_1} + ... + x_{i_{n+1}}}{n+1}\right) x_{i_j} \end{aligned}$$

for all $k,j\in\{1,...,n+1\}$, we deduce that $A=0.$
On the other hand, since

$$\begin{aligned}
&\frac{1}{P_H^{n+1}}\sum_{i_1,\dots,i_{n+1}\in H}p_{i_1}...p_{i_{n+1}}f'_+\left(\frac{x_{i_1}+...+x_{i_n}}{n}\right)(x_{i_k}-x_{i_{n+1}})\\
=&\frac{1}{P_H^{n+1}}\sum_{i_1,\dots,i_{n+1}\in H}p_{i_1}...p_{i_{n+1}}f'_+\left(\frac{x_{i_1}+...+x_{i_n}}{n}\right)(x_{i_1}-x_{i_{n+1}})
\end{aligned}$$

for all $k\in\{1,...,n\}$, we get that

$$\begin{aligned}
B &= \frac{1}{n+1}\cdot\frac{1}{P_H^{n+1}}\sum_{i_1,\dots,i_{n+1}\in H}p_{i_1}...p_{i_{n+1}}f'_+\left(\frac{x_{i_1}+...+x_{i_n}}{n}\right)(x_{i_1}-x_{i_{n+1}})\\
&= \frac{1}{n+1}\left[\frac{1}{P_H^{n}}\sum_{i_1,\dots,i_{n}\in H}p_{i_1}...p_{i_{n}}f'_+\left(\frac{x_{i_1}+...+x_{i_n}}{n}\right)x_{i_1}\right.\\
&\quad\left.-\frac{1}{P_H}\sum_{i\in H}p_ix_i\frac{1}{P_H^{n}}\sum_{i_1,\dots,i_{n}\in H}p_{i_1}...p_{i_{n}}f'_+\left(\frac{x_{i_1}+...+x_{i_n}}{n}\right)\right]
\end{aligned}$$

and by (**??**), the inequality (**??**) is proved.
Now, if we choose in (**??**):

$$x=\frac{x_{i_1}+...+x_{i_n}}{n}\in\mathring{I}\text{ and }y=\frac{1}{P_H}\sum_{i\in H}p_ix_i\in\mathring{I},$$

where $i_1,...,i_n\in H$, we deduce:

$$\begin{aligned}
&f'_+\left(\frac{1}{P_H}\sum_{i\in H}p_ix_i\right)\left[\frac{x_{i_1}+...+x_{i_n}}{n}-\frac{1}{P_H}\sum_{i\in H}p_ix_i\right]\\
\leq\;& f\left(\frac{x_{i_1}+...+x_{i_n}}{n}\right)-f\left(\frac{1}{P_H}\sum_{i\in H}p_ix_i\right)\\
\leq\;& f'_+\left(\frac{x_{i_1}+...+x_{i_n}}{n}\right)\left[\frac{x_{i_1}+...+x_{i_n}}{n}-\frac{1}{P_H}\sum_{i\in H}p_ix_i\right]
\end{aligned}\tag{2.6}$$

for all $i_1,...,i_n\in H$.
If we multiply the inequality (**??**) by $p_{i_1}...p_{i_n}\geq 0$ and summing over $i_1,...,i_n\in H$

we deduce:

$$
\begin{aligned}
C &: = \frac{1}{P_H^n}\sum_{i_1,\dots,i_n\in H} p_{i_1}\dots p_{i_n} f'_+\left(\frac{1}{P_H}\sum_{i\in H} p_i x_i\right) \\
&\quad\times\left(\frac{x_{i_1}+\dots+x_{i_n}}{n}-\frac{1}{P_H}\sum_{i\in H}p_i x_i\right) \\
&\leq \mathfrak{I}_n\left(f,p,H,x\right)-f\left(\frac{1}{P_H}\sum_{i\in H}p_i x_i\right) \\
&\leq \frac{1}{P_H^n}\sum_{i_1,\dots,i_n\in H} p_{i_1}\dots p_{i_n} f'_+\left(\frac{x_{i_1}+\dots+x_{i_n}}{n}\right) \\
&\quad\times\left(\frac{x_{i_1}+\dots+x_{i_n}}{n}-\frac{1}{P_H}\sum_{i\in H}p_i x_i\right) \\
&: = D.
\end{aligned}
\tag{2.7}
$$

As it is clear that:

$$
\frac{1}{P_H^n}\sum_{i_1,\dots,i_n\in H} p_{i_1}\dots p_{i_n} x_{i_k}=\frac{1}{P_H}\sum_{i\in H}p_i x_i
$$

for all $k\in\{1,\dots,n\}$, then $C=0$.
In addition, because

$$
\begin{aligned}
&\frac{1}{P_H^n}\sum_{i_1,\dots,i_n\in H} p_{i_1}\dots p_{i_n} f'_+\left(\frac{x_{i_1}+\dots+x_{i_n}}{n}\right)x_{i_k} \\
= &\frac{1}{P_H^n}\sum_{i_1,\dots,i_n\in H} p_{i_1}\dots p_{i_n} f'_+\left(\frac{x_{i_1}+\dots+x_{i_n}}{n}\right)x_{i_1}
\end{aligned}
$$

for all $k\in\{1,\dots,n\}$, then

$$
\begin{aligned}
D &= \frac{1}{P_H^n}\sum_{i_1,\dots,i_n\in H} p_{i_1}\dots p_{i_n} f'_+\left(\frac{x_{i_1}+\dots+x_{i_n}}{n}\right)x_{i_1} \\
&\quad -\frac{1}{P_H}\sum_{i\in H}p_i x_i\frac{1}{P_H^n}\sum_{i_1,\dots,i_n\in H} p_{i_1}\dots p_{i_n} f'_+\left(\frac{x_{i_1}+\dots+x_{i_n}}{n}\right).
\end{aligned}
$$

Consequently, by (**??**) we deduce the inequality (**??**) and the theorem is thus proved. ▮

Remark 1. *From the above theorem we have the inequalities:*

$$
\begin{aligned}
f\left(\frac{1}{P_H}\sum_{i\in H}p_i x_i\right) &\leq \dots\leq \mathfrak{I}_{n+1}\left(f,p,H,x\right)\leq \mathfrak{I}_n\left(f,p,H,x\right) \\
&\leq \dots\leq \mathfrak{I}_1\left(f,p,H,x\right)=\frac{1}{P_H}\sum_{i\in H}p_i f\left(x_i\right)
\end{aligned}
\tag{2.8}
$$

which have been established for the first time by S. S. Dragomir and J. E. Pečarić in 1987 [**?**] *for the general case of convex mappings* $f: C\subseteq X\longrightarrow\mathbb{R}$ *defined on convex subsets from linear spaces. Note that, in the paper* [**?**] *the authors used*

Jensen's discrete inequality for multiple sums, i.e., a different argument that we used before.

In what follows we shall estimate the sequence of functionals E_n.

Theorem 6. *With the assumptions of Theorem ??, and the conditions that $s>1$ and $\frac{1}{s}+\frac{1}{p}=1$, we have the inequality:*

$$
\begin{aligned}
(2.9)\quad 0 &\le E_n(f,p,H,x) \\
&\le \left(\frac{1}{P_H^n}\sum_{i_1,\dots,i_n\in H} p_{i_1}\dots p_{i_n}\left|f'_+\left(\frac{x_{i_1}+\dots+x_{i_n}}{n}\right)\right|^s\right)^{\frac{1}{s}} \\
&\quad\times\left(\frac{1}{P_H^n}\sum_{i_1,\dots,i_n\in H} p_{i_1}\dots p_{i_n}\left|\frac{x_{i_1}+\dots+x_{i_n}}{n}-\frac{1}{P_H}\sum_{i\in H}p_i x_i\right|^q\right)^{\frac{1}{q}}
\end{aligned}
$$

for all $(f,p,H,x)\in Conv(I)\times S_+^*(\mathbb{R})\times\mathfrak{P}_f^*(\mathbb{N})\times S\left(\mathring{I}\right)$ and $n\ge 1$.

Proof. Using an identity from Theorem **??** we have that

$$
\begin{aligned}
0 &\le E_n(f,p,H,x)=|E_n(f,p,H,x)| \\
&\le \frac{1}{P_H^n}\sum_{i_1,\dots,i_n\in H} p_{i_1}\dots p_{i_n}\left|f'_+\left(\frac{x_{i_1}+\dots+x_{i_n}}{n}\right)\right|\left|\frac{x_{i_1}+\dots+x_{i_n}}{n}-\frac{1}{P_H}\sum_{i\in H}p_i x_i\right|.
\end{aligned}
$$

Applying Hölder's inequality for multiple sums, i.e.,

$$
\begin{aligned}
&\left|\sum_{i_1,\dots,i_n\in H} p_{i_1,\dots,i_n}\ a_{i_1,\dots,i_n}\ b_{i_1,\dots,i_n}\right| \\
&\le \left(\sum_{i_1,\dots,i_n\in H} p_{i_1,\dots,i_n}\ |a_{i_1,\dots,i_n}|^s\right)^{\frac{1}{s}}\times\left(\sum_{i_1,\dots,i_n\in H} p_{i_1,\dots,i_n}\ |b_{i_1,\dots,i_n}|^q\right)^{\frac{1}{q}},
\end{aligned}
$$

where $\frac{1}{s}+\frac{1}{q}=1$, $s>1$, $a_{i_1,\dots,i_n}\ b_{i_1,\dots,i_n}\in\mathbb{R}$, $p_{i_1,\dots,i_n}>0$ for all $i_1,\dots,i_n\in H$, we deduce the desired inequality (**??**). ∎

The above theorem admits the following corollary which is more useful in applications.

Corollary 1. *With the above assumptions and if the condition*

$$M:=\sup_{x\in\mathring{I}}\left|f'_+(x)\right|<\infty,$$

holds, then we have the estimate:

$$0\le E_n(f,p,H,x)\le\frac{\sqrt{2}M}{2\sqrt{n}P_H}\left[\sum_{i,j\in H}p_i p_j(x_i-x_j)^2\right]^{\frac{1}{2}} \tag{2.10}$$

for all $n\ge 1$ and $(p,H,x)\in S_+^*(\mathbb{R})\times\mathfrak{P}_f^*(\mathbb{N})\times S\left(\mathring{I}\right)$.

Proof. For $s=p=2$ and by the boundedness of $f_{+}^{\prime}$ on $\mathring{I}$ we get:

$$E_{n}\left(f,p,H,x\right)\leq M\left(\frac{1}{P_{H}^{n}}\sum_{i_{1},\dots,i_{n}\in H}p_{i_{1}}...p_{i_{n}}\left|\frac{x_{i_{1}}+...+x_{i_{n}}}{n}-\frac{1}{P_{H}}\sum_{i\in H}p_{i}x_{i}\right|^{2}\right)^{\frac{1}{2}}.$$

Let us compute:

$$\begin{aligned} F &:= \frac{1}{P_{H}^{n}}\sum_{i_{1},\dots,i_{n}\in H}p_{i_{1}}...p_{i_{n}}\left|\frac{x_{i_{1}}+...+x_{i_{n}}}{n}-\frac{1}{P_{H}}\sum_{i\in H}p_{i}x_{i}\right|^{2} \\ &= \frac{1}{P_{H}^{n}}\sum_{i_{1},\dots,i_{n}\in H}p_{i_{1}}...p_{i_{n}} \\ &\quad\times\left|\left(\frac{x_{i_{1}}+...+x_{i_{n}}}{n}\right)^{2}-2\left(\frac{1}{P_{H}}\sum_{i\in H}p_{i}x_{i}\right)\left(\frac{x_{i_{1}}+...+x_{i_{n}}}{n}\right)\right. \\ &\quad\left.+\left(\frac{1}{P_{H}}\sum_{i\in H}p_{i}x_{i}\right)^{2}\right|. \end{aligned}$$

We also have:

$$\left(x_{i_{1}}+...+x_{i_{n}}\right)^{2}=\sum_{j=1}^{n}x_{i_{j}}^{2}+2\sum_{1\leq j<l\leq n}x_{i_{j}}x_{i_{l}},$$

then

$$\begin{aligned} &\frac{1}{P_{H}^{n}}\sum_{i_{1},\dots,i_{n}\in H}p_{i_{1}}...p_{i_{n}}\left(\frac{x_{i_{1}}+...+x_{i_{n}}}{n}\right)^{2} \\ &= \frac{1}{n^{2}}\cdot\frac{1}{P_{H}^{n}}\sum_{i_{1},\dots,i_{n}\in H}p_{i_{1}}...p_{i_{n}}\left[\sum_{j=1}^{n}x_{i_{j}}^{2}+2\sum_{1\leq j<l\leq n}x_{i_{j}}x_{i_{l}}\right] \\ &= \frac{1}{n^{2}}\left[n\cdot\frac{1}{P_{H}}\sum_{i\in H}p_{i}x_{i}^{2}+2\cdot\frac{n\left(n-1\right)}{2}\left(\frac{1}{P_{H}}\sum_{i\in H}p_{i}x_{i}\right)^{2}\right] \\ &= \frac{1}{n}\left[\frac{1}{P_{H}}\sum_{i\in H}p_{i}x_{i}^{2}+\left(n-1\right)\left(\frac{1}{P_{H}}\sum_{i\in H}p_{i}x_{i}\right)^{2}\right]. \end{aligned}$$

On the other hand we have:

$$\frac{1}{P_{H}^{n}}\sum_{i_{1},\dots,i_{n}\in H}p_{i_{1}}...p_{i_{n}}\left(\frac{x_{i_{1}}+...+x_{i_{n}}}{n}\right)=\frac{1}{P_{H}}\sum_{i\in H}p_{i}x_{i}.$$

Thus,

$$\begin{aligned} F &= \frac{1}{n}\left[\frac{1}{P_H}\sum_{i\in H}p_ix_i^2+(n-1)\left(\frac{1}{P_H}\sum_{i\in H}p_ix_i\right)^2\right]-2\left(\frac{1}{P_H}\sum_{i\in H}p_ix_i\right)^2 \\ &\quad +\left(\frac{1}{P_H}\sum_{i\in H}p_ix_i\right)^2 \\ &= \frac{1}{n}\left[\frac{1}{P_H}\sum_{i\in H}p_ix_i^2-\left(\frac{1}{P_H}\sum_{i\in H}p_ix_i\right)^2\right] \\ &= \frac{1}{nP_H^2}\left[\sum_{i\in H}p_i\sum_{i\in H}p_ix_i^2-\left(\sum_{i\in H}p_ix_i\right)^2\right] \\ &= \frac{1}{2nP_H^2}\sum_{i,j\in H}p_ip_j\left(x_i-x_j\right)^2 \end{aligned}$$

and the corollary is proved. ∎

We have the following corollary of Theorem **??**:

Corollary 2. *With the assumptions of Corollary* **??** *we have the estimate:*

$$\begin{aligned} 0 &\leq \mathfrak{J}_n(f,p,H,x)-\mathfrak{J}_{n+1}(f,p,H,x) \\ &\leq \frac{\sqrt{2}M}{2(n+1)\sqrt{n}}\cdot\frac{1}{P_H}\left[\sum_{i,j\in H}p_ip_j\left(x_i-x_j\right)^2\right]^{\frac{1}{2}} \end{aligned} \tag{2.11}$$

$$\begin{aligned} 0 &\leq \mathfrak{J}_n(f,p,H,x)-f\left(\frac{1}{P_H}\sum_{i\in H}p_ix_i\right) \\ &\leq \frac{\sqrt{2}M}{2\sqrt{n}P_H}\left[\sum_{i,j\in H}p_ip_j\left(x_i-x_j\right)^2\right]^{\frac{1}{2}} \end{aligned} \tag{2.12}$$

for all $(p,H,x)\in S_+^*(\mathbb{R})\times\mathfrak{P}_f^*(\mathbb{N})\times S\left(\mathring{I}\right)$.

We can conclude with the following theorems of convergence:

Theorem 7. *(Convergence) If* $f: I\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ *is a convex function having a bounded derivative on* $\mathring{I}$*, then for all* $(p,H,x)\in S_+^*(\mathbb{R})\times\mathfrak{P}_f^*(\mathbb{N})\times S\left(\mathring{I}\right)$ *we have:*

$$\lim_{n\longrightarrow\infty}n^{\alpha}\left[\mathfrak{J}_n(f,p,H,x)-\mathfrak{J}_{n+1}(f,p,H,x)\right]=0 \textit{ for } \alpha\in\left[0,\frac{3}{2}\right)$$

and

$$\lim_{n\longrightarrow\infty}n^{\beta}\left[\mathfrak{J}_n(f,p,H,x)-f\left(\frac{1}{P_H}\sum_{i\in H}p_ix_i\right)\right]=0 \textit{ for } \beta\in\left[0,\frac{1}{2}\right)$$

The proof is obvious by the inequalities (**??**) and (**??**).

Finally, we have the following.

Theorem 8. *(Uniform Convergence) If $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a convex function having a bounded derivative on $\mathring{I}$, then*

$$\lim_{n\longrightarrow\infty} n^{\alpha}\left[\mathfrak{J}_n(f,p,H,x)-\mathfrak{J}_{n+1}(f,p,H,x)\right]=0 \text{ for } \alpha\in\left[0,\frac{3}{2}\right)$$

and

$$\lim_{n\longrightarrow\infty} n^{\beta}\left[\mathfrak{J}_n(f,p,H,x)-f\left(\frac{1}{P_H}\sum_{i\in H}p_ix_i\right)\right]=0 \text{ for } \beta\in\left[0,\frac{1}{2}\right)$$

uniformly on $S_+^*(\mathbb{R})\times\mathfrak{P}_f^*(\mathbb{N})\times S([a,b])$ where $[a,b]\subset\mathring{I}$.

Proof. Let $(p,H,x)\in S_+^*(\mathbb{R})\times\mathfrak{P}_f^*(\mathbb{N})\times S([a,b])$. Then $|x_i-x_j|\le b-a$ $(i,j\in H)$ and, by (**??**) and (**??**) we get that:

$$0\le\mathfrak{J}_n(f,p,H,x)-\mathfrak{J}_{n+1}(f,p,H,x)\le\frac{\sqrt{2}M(b-a)}{2(n+1)\sqrt{n}}$$

and

$$0\le\mathfrak{J}_n(f,p,H,x)-f\left(\frac{1}{P_H}\sum_{i\in H}p_ix_i\right)\le\frac{\sqrt{2}M(b-a)}{2\sqrt{n}},$$

which proves the theorem. ∎

Remark 2. *Similar results can be stated for differentiable mappings of several variables by using the inequality*

$$\langle\nabla f(y),x-y\rangle\le f(x)-f(y)\le\langle\nabla f(x),x-y\rangle$$

for any $x,y\in\mathbb{R}^m$. We omit the details.

3. Applications for the Mapping $\log_b$

If we assume that $g : I\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ is a concave mapping on I and $(p,H,x)\in S_+^*(\mathbb{R})\times\mathfrak{P}_f^*(\mathbb{N})\times S\left(\mathring{I}\right)$ we have that:

$$\begin{aligned}\mathfrak{J}_1(g,p,H,x)&\le\cdots\le\mathfrak{J}_n(g,p,H,x)\\&\le\mathfrak{J}_{n+1}(g,p,H,x)\le\cdots\le g\left(\frac{1}{P_H}\sum_{i\in H}p_ix_i\right)\end{aligned}\tag{3.1}$$

and

$$0\le\mathfrak{J}_{n+1}(g,p,H,x)-\mathfrak{J}_n(g,p,H,x)\le\frac{1}{n+1}L_n(g,p,H,x)\tag{3.2}$$

and

$$0\le g\left(\frac{1}{P_H}\sum_{i\in H}p_ix_i\right)-\mathfrak{J}_n(g,p,H,x)\le L_n(g,p,H,x)\tag{3.3}$$

respectively, where:

$$\begin{aligned}L_n(g,p,H,x)\ &:=\frac{1}{P_H}\sum_{i\in H}p_ix_i\cdot\frac{1}{P_H^n}\sum_{i_1,\dots,i_n\in H}p_{i_1}...p_{i_n}g'_+\left(\frac{x_{i_1}+...+x_{i_n}}{n}\right)\\&\quad-\frac{1}{P_H^n}\sum_{i_1,\dots,i_n\in H}p_{i_1}...p_{i_n}g'_+\left(\frac{x_{i_1}+...+x_{i_n}}{n}\right)x_{i_1}.\end{aligned}$$

If $M := \sup_{x\in\mathring{I}} \left|g'_+(x)\right| < \infty$, then we have:

(3.4)

$$0 \le \mathfrak{J}_{n+1}(g,p,H,x) - \mathfrak{J}_n(g,p,H,x) \le \frac{\sqrt{2}M}{2(n+1)\sqrt{n}} \cdot \frac{1}{P_H}\left[\sum_{i,j\in H} p_i p_j (x_i - x_j)^2\right]^{\frac{1}{2}}$$

and

(3.5)

$$0 \le g\left(\frac{1}{P_H}\sum_{i\in H} p_i x_i\right) - \mathfrak{J}_n(g,p,H,x) \le \frac{\sqrt{2}M}{2\sqrt{n}} \cdot \frac{1}{P_H}\left[\sum_{i,j\in H} p_i p_j (x_i - x_j)^2\right]^{\frac{1}{2}}.$$

If $x_i \in [a,b] \subset \mathring{I}$, $i \in H$, we have that:

$$0 \le \mathfrak{J}_{n+1}(g,p,H,x) - \mathfrak{J}_n(g,p,H,x) \le \frac{\sqrt{2}M(b-a)}{2(n+1)\sqrt{n}} \tag{3.6}$$

and

$$0 \le g\left(\frac{1}{P_H}\sum_{i\in H} p_i x_i\right) - \mathfrak{J}_n(g,p,H,x) \le \frac{\sqrt{2}M(b-a)}{2\sqrt{n}}. \tag{3.7}$$

Now, for the concave mapping $g : (0,\infty) \longrightarrow \mathbb{R}$, $g(x) = \log_b x$, $b > 1$, we define the sequence of mappings:

$$H_n(p,x,b) := \frac{1}{P_s^n}\sum_{i_1,\dots,i_n=1}^{s} p_{i_1}\dots p_{i_n} \log_b\left(\frac{x_{i_1}+\dots+x_{i_n}}{n}\right),\ n \ge 1$$

where $s \ge 1$ is given in $\mathbb{N}$ and $p_i, x_i > 0$ for all $i \in \mathbb{N}$, $i \ge 1$ and $P_s := \sum_{i=1}^{s} p_i$.

Note that for this case we have:

$$\begin{aligned}
0 &\le L_n(p,x,b) \\
&= \frac{n}{\ln b}\left[\frac{1}{P_s}\sum_{i=1}^{s} p_i x_i \frac{1}{P_s^n}\sum_{i_1,\dots,i_n=1}^{s}\frac{p_{i_1}\dots p_{i_n}}{x_{i_1}+\dots+x_{i_n}}\right. \\
&\quad \left. - \frac{1}{P_s^n}\sum_{i_1,\dots,i_n=1}^{s}\frac{p_{i_1}\dots p_{i_n}}{x_{i_1}+\dots+x_{i_n}} x_{i_1}\right] \\
&= \frac{n}{P_s^{n+1}\ln b}\sum_{i_1,\dots,i_{n+1}=1}^{s}\frac{p_{i_1}\dots p_{i_{n+1}}\left(x_{i_{n+1}}-x_{i_1}\right)}{x_{i_1}+\dots+x_{i_n}} \\
&= \frac{n}{P_s^n\ln b}\sum_{i_1,\dots,i_n=1}^{s}\frac{p_{i_1}\dots p_{i_n}}{x_{i_1}+\dots+x_{i_n}}\left(\frac{1}{P_s}\sum_{i=1}^{s} p_i x_i - \frac{x_{i_1}+\dots+x_{i_n}}{n}\right) \\
&= \frac{n}{P_s^n\ln b}\sum_{i_1,\dots,i_n=1}^{s}\frac{p_{i_1}\dots p_{i_n}}{x_{i_1}+\dots+x_{i_n}}\left(\frac{1}{P_s}\sum_{i=1}^{s} p_i x_i - x_{i_1}\right).
\end{aligned}$$

For $n=1$ we have

$$\begin{aligned} L_1(p,x,b) &= \frac{1}{\ln b P_s}\sum_{j=1}^{s}\frac{p_j}{x_j}\left(\frac{1}{P_s}\sum_{i=1}^{s}p_ix_i - x_j\right) \\ &= \frac{1}{\ln b}\left[\frac{1}{P_s}\sum_{i=1}^{s}p_ix_i\cdot\frac{1}{P_s}\sum_{i=1}^{s}\frac{p_i}{x_i}-1\right] \end{aligned}$$

On the other hand we have:

$$\frac{1}{2P_s^2}\sum_{i,j=1}^{s}\frac{p_ip_j}{x_ix_j}(x_i-x_j)^2 = \frac{1}{P_s^2}\sum_{i=1}^{s}p_ix_i\sum_{i=1}^{s}\frac{p_i}{x_i}-1$$

and then:

$$L_1(p,x,b) = \frac{1}{2\ln b}\sum_{i,j=1}^{s}\frac{p_ip_j}{x_ix_j}(x_i-x_j)^2.$$

If $x_i \geq m > 0$, $i\in\{1,...,s\}$, then we have:

$$g'(x) = \frac{1}{\ln b}\cdot\frac{1}{x}\leq\frac{1}{\ln b}\cdot\frac{1}{m},\ x\in[m,\infty).$$

By the inequality (**??**) we have:

(3.8)

$$0\leq H_{n+1}(p,x,b)-H_n(p,x,b)\leq\frac{1}{m(n+1)\sqrt{n}}\cdot\frac{1}{P_s}\left[\sum_{1\leq i<j\leq s}p_ip_j(x_i-x_j)^2\right]^{\frac{1}{2}},$$

and by (**??**) we derive

(3.9)

$$0\leq\log_b\left(\frac{1}{P_s}\sum_{i=1}^{s}p_ix_i\right)-H_n(p,x,b)\leq\frac{1}{m\sqrt{n}}\cdot\frac{1}{P_s}\left[\sum_{1\leq i<j\leq s}p_ip_j(x_i-x_j)^2\right]^{\frac{1}{2}}$$

for all $p\in S_+^*(\mathbb{R})$, $b>1$ and $s\geq 1$.

Finally, if $x_i\in[m,M]\subset(0,\infty)$, then we have:

$$0\leq H_{n+1}(p,x,b)-H_n(p,x,b)\leq\frac{\sqrt{2}(M-m)}{2m(n+1)\sqrt{n}} \tag{3.10}$$

and

$$0\leq\log_b\left(\frac{1}{P_s}\sum_{i=1}^{s}p_ix_i\right)-H_n(p,x,b)\leq\frac{\sqrt{2}(M-m)}{2m\sqrt{n}} \tag{3.11}$$

for all $s\geq 1$, for all $x_i\in[m,M]\subset(0,\infty)$, $i\in\{1,...,s\}$, $b>1$, $n\geq 1$ and $p_i>0$ for all $i\in\mathbb{N}$.

These two inequalities enable us to state the following results of convergence:

$$\lim_{n\longrightarrow\infty}n^{\alpha}\left[H_{n+1}(p,x,b)-H_n(p,x,b)\right]=0 \text{ for } \alpha\in\left[0,\frac{3}{2}\right)$$

and

$$\lim_{n\longrightarrow\infty}n^{\beta}\left[\log_b\left(\frac{1}{P_s}\sum_{i=1}^{s}p_ix_i\right)-H_n(p,x,b)\right]=0 \text{ for } \beta\in\left[0,\frac{1}{2}\right)$$

uniformly for $(p, s, x, b) \in S_+^* (\mathbb{R}) \times \mathbb{M}^* \times S[m, M] \times (1, \infty)$.

4. Some Estimation for the $n - b-$Entropy Mapping

Suppose that X is a discrete random variable whose range is finite $R = \{x_1, ..., x_s\}$. Let $p_i = P\{X = x_i\}$.
We can introduce the following concept.

Definition 1. *The $n - b-$entropy mapping associated to the random variable X can be defined by:*

$$H_n(X; b) := \sum_{i_1,...,i_n=1}^{s} p_{i_1}...p_{i_n} \log_b \left(\frac{\frac{1}{p_{i_1}} + ... + \frac{1}{p_{i_n}}}{n} \right) \tag{4.1}$$

where $n \geq 1$.

Note that for $n = 1$ we recapture the usual b-entropy, i.e.,

$$H_n(X; b) = H(X, b) = \sum_{i=1}^{s} p_i \log_b \frac{1}{p_i}.$$

Using the results from section 3, we have

$$\begin{aligned} 0 &\leq H_1(X; b) \leq H_2(X; b) \leq ... \leq H_n(X; b) \\ &\leq H_{n+1}(X; b) \leq ... \leq \log_b s \quad \text{for all } n \geq 1; \end{aligned} \tag{4.2}$$

and

$$0 \leq \log_b r - H_n(X; b) \leq h_n(X; b) \quad \text{for all } n \geq 1 \tag{4.3}$$

and

$$0 \leq H_{n+1}(X; b) - H_n(X; b) \leq \frac{1}{n+1} h_n(X; b) \quad \text{for all } n \geq 1 \tag{4.4}$$

where:

$$\begin{aligned} h_n(X; b) &= \frac{n}{\ln b} \sum_{i_1,...,i_n=1}^{s} \frac{p_{i_1}...p_{i_n}}{\frac{1}{p_{i_1}} + ... + \frac{1}{p_{i_n}}} \left(\sum_{i=1}^{s} p_i \cdot \frac{1}{p_i} - \frac{\frac{1}{p_{i_1}} + ... + \frac{1}{p_{i_n}}}{n} \right) \\ &= \frac{n}{\ln b} \sum_{i_1,...,i_n=1}^{s} \frac{p_{i_1}...p_{i_n}}{\frac{1}{p_{i_1}} + ... + \frac{1}{p_{i_n}}} \left(s - \frac{\frac{1}{p_{i_1}} + ... + \frac{1}{p_{i_n}}}{n} \right) \\ &= \frac{n}{\ln b} \left[s \sum_{i_1,...,i_n=1}^{s} \frac{p_{i_1}...p_{i_n}}{\frac{1}{p_{i_1}} + ... + \frac{1}{p_{i_n}}} - \frac{1}{n} \right]. \end{aligned}$$

For $n = 1$, we have:

$$\begin{aligned} h_1(X; b) &= \frac{1}{2 \ln b} \sum_{i,j=1}^{s} \frac{p_i p_j}{\frac{1}{p_i} \cdot \frac{1}{p_j}} \left(\frac{1}{p_i} - \frac{1}{p_j} \right)^2 \\ &= \frac{1}{2 \ln b} \sum_{i,j=1}^{s} (p_i - p_j)^2 \\ &= \frac{1}{\ln b} \sum_{1 \leq i < j \leq s} (p_i - p_j)^2. \end{aligned}$$

Let us denote $p_M := \max_{i=\overline{1,s}} \{p_i\}$, $p_m := \min_{i=\overline{1,s}} \{p_i\}$. Then $\frac{1}{p_i} \in \left[\frac{1}{p_M}, \frac{1}{p_m}\right]$, and by (**??**) and (**??**) we have that:

$$\begin{aligned} (4.5) \qquad 0 &\leq H_{n+1}(X;b) - H_n(X;b) \\ &\leq \frac{p_M}{(n+1)\sqrt{n}} \left[\sum_{1\leq i<j\leq s}^{s} \frac{(p_i-p_j)^2}{p_i p_j}\right]^{\frac{1}{2}} \\ &\leq \frac{1}{(n+1)\sqrt{n}} \left[\sum_{1\leq i<j\leq s}^{s} \frac{(p_i-p_j)^2}{p_i p_j}\right]^{\frac{1}{2}}, \quad n \geq 1 \end{aligned}$$

and

$$\begin{aligned} (4.6) \qquad 0 &\leq \log_b s - H_n(X;b) \\ &\leq \frac{p_M}{\sqrt{n}} \left[\sum_{1\leq i<j\leq s}^{s} \frac{(p_i-p_j)^2}{p_i p_j}\right]^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{n}} \left[\sum_{1\leq i<j\leq s}^{s} \frac{(p_i-p_j)^2}{p_i p_j}\right]^{\frac{1}{2}}. \end{aligned}$$

Now, if we use the inequalities (**??**) and (**??**) we have that:

$$(4.7) \qquad 0 \leq H_{n+1}(X;b) - H_n(X;b) \leq \frac{\sqrt{2}(p_M - p_m)}{2p_m(n+1)\sqrt{n}}, \ n \geq 1$$

and

$$(4.8) \qquad 0 \leq \log_b s - H_n(X;b) \leq \frac{\sqrt{2}(p_M - p_m)}{2p_m(n+1)\sqrt{n}}, \ n \geq 1.$$

The following result of convergence also holds:

$$(4.9) \qquad \lim_{n\to\infty} n^{\alpha} [H_{n+1}(X;b) - H_n(X;b)] = 0, \ \alpha \in \left[0, \frac{3}{2}\right)$$

and

$$(4.10) \qquad \lim_{n\to\infty} n^{\beta} [\log_b s - H_n(X;b)] = 0, \ \beta \in \left[0, \frac{1}{2}\right)$$

uniformly by rapport of $b > 1$ and $s \geq 1$.

We can state the following proposition.

Proposition 1. *With the above assumptions we have the estimation:*

$$(4.11) \qquad 0 \leq \log_b s - H_b(X) \leq \frac{1}{\ln b} \sum_{1\leq i<j\leq s}^{s} (p_i - p_j)^2.$$

Proof. Follows from the inequality (**??**) taking into account that

$$h_1(X;b) = \frac{1}{\ln b} \sum_{1\leq i<j\leq s}^{s} (p_i - p_j)^2.$$

∎

Remark 3. *The result we obtained above is exactly Theorem 4.3 from [?] written in another form.*

Corollary 3. *With the above assumptions and if*

$$\max_{1\le i<j\le s} |p_i - p_j| \le \sqrt{\frac{2\varepsilon \ln s}{s(s-1)}}$$

for $\varepsilon > 0$, *then:*

$$0 \le \log_b s - H_b(X) \le \varepsilon. \tag{4.12}$$

The following theorem holds.

Theorem 9. *Let* $\varepsilon > 0$. *If* $p_i > 0$ $\left(i = \overline{1,s}\right)$ $(s \ge 2)$ *are such that:*

$$\frac{p_i}{p_j} \in \left[\frac{2+q-\sqrt{q(q+4)}}{2}, \frac{2+q+\sqrt{q(q+4)}}{2}\right],\ 1 \le i < j \le s$$

where:

$$q = \frac{2\varepsilon^2 n(n+1)^2}{s(s-1)},\ n \ge 1.$$

Then we have the estimate:

$$0 \le H_{n+1}(X;b) - H_n(X;b) \le \varepsilon,\ n \ge 1. \tag{4.13}$$

Proof. By the inequality (**??**) we have that

$$0 \le H_{n+1}(X;b) - H_n(X;b) \le \frac{1}{(n+1)\sqrt{n}} \left[\sum_{1\le i<j\le s}^{s} \frac{(p_i-p_j)^2}{p_i p_j}\right]^{\frac{1}{2}}.$$

Let us consider the inequality:

$$\frac{(p_i-p_j)^2}{p_i p_j} \le q,\ 1 \le i < j \le s,\ q > 0,$$

which is equivalent to

$$p_i^2 - (2+q)\, p_i p_j + p_i^2 \le 0,\ 1 \le i < j \le s,\ q > 0$$

or

$$\left(\frac{p_i}{p_j}\right)^2 - (2+q)\frac{p_i}{p_j} + 1 \le 0,\ 1 \le i < j \le s,$$

or, additionally,

$$\frac{p_i}{p_j} \in \left[\frac{2+q-\sqrt{q(q+4)}}{2}, \frac{2+q+\sqrt{q(q+4)}}{2}\right],\ 1 \le i < j \le s.$$

Now, if we choose

$$q = \frac{2\varepsilon^2 n(n+1)^2}{s(s-1)},$$

then

$$\sum_{1\le i<j\le s}^{s}\frac{(p_i-p_j)^2}{p_ip_j}\le\frac{s(s-1)}{2}q=\varepsilon^2 n(n+1)^2$$

and

$$0\le H_{n+1}(X;b)-H_n(X;b)\le\frac{1}{(n+1)\sqrt{n}}\left[\varepsilon^2 n(n+1)^2\right]^{\frac{1}{2}}=\varepsilon.$$

The theorem is thus proved. ∎

Similarly, we can prove the following theorem too.

Theorem 10. *Let $\varepsilon>0$. If $p_i>0$ $\left(i=\overline{1,s}\right)$ $(s\ge 2)$ are such that:*

$$\frac{p_i}{p_j}\in\left[\frac{2+q-\sqrt{q(q+4)}}{2},\frac{2+q+\sqrt{q(q+4)}}{2}\right],\ 1\le i<j\le s,$$

where

$$q=\frac{2\varepsilon^2 n}{s(s-1)},$$

then we have the estimate;

$$0\le\log_b s-H_b(X)\le\varepsilon,\ n\ge 1.$$

The proof goes by the inequality (**??**) and we shall omit the details.

References

[1] S. S. Dragomir and N.M. Ionescu, Some converse of Jensen's inequality and applications, *Anal. Numer. Theor Approx.* (Cluj-Napoca), **23**(1994),71-78.

[2] S. S. Dragomir and C. J. Goh, A counterpart of Jensen's discrete inequality for differentiable convex mappings and applications in Information Theory, *Math. Comp. Modelling,* **24**(2)(1996), 1-11.

[3] S. S. Dragomir and C. J. Goh, A counterpart of Jensen's continuous inequality and applications in information theory, *Intern. Journal of Math. and Math. Sciences (in press).*

[4] S. S. Dragomir and G. Keady, A Hadamard-Jensen inequality for convex functions to higher dimension and an application to the elastic torsion problem, UWA, *Research Report*, June 1996. http://maths.uwa.edu.au/~keady/papers.html.

[5] M. Matić, C. E. M. Pearce and J. Pečarić, Improvements of some bounds on entropy measures in information theory, *Math. Inequal. and Appl.*, **1**(1998) 295–304.

[6] M. Matić, Ph. D. Thesis, *Inequalities of Jensen's type with applications in Information theory*, (*Nejednakosti Jensenovog tipa s primjenama u teoriji informacija* - in Croatian), University of Zagreb.

[7] M. Matić, C. E. M. Pearce and J. Pečarić, Refinements of some bounds in information theory, *J. Austr. Math. Soc. Ser.* B, to appear.

[8] R. J. McEliece, *The Theory of Information and Coding*, Addison-Wesley, Reading, Mass. 1977.

[9] S.S. Dragomir, V. Glušc̆ević, C.E.M. Pearce and J. Pečarić, New inequalities for logarithmic map and their application for entropy and mutual information, *submitted. http://matilda.vu.edu.au/~rgmia/severdvi/logmap4.dvi*

[9] V. Glušc̆ević, S. S. Dragomir and. J. Sunde, Inequalities and applications in information theory, *submitted.*

[10] S.S. Dragomir, J.Sunde and M. Scholz, Some upper bounds for the relative entropy and applications, *submitted. http://matilda.vu.edu.au/~rgmia/severdvi/*SSJS33.dvi

[11] J.E. Pecaric and S.S. Dragomir, A refinement of Jensen's inequality with applications, *Studia Univ. Babes-Bolyai, Math.,* **34** (1)(1989), 15-19.

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MCMC, 8001 Victoria, Australia.
E-mail address: sever@matilda.vu.edu.au
URL: http://matilda.vu.edu.au/~rgmia/dragomirweb.html