ON TORICELLI'S PROBLEM IN INNER PRODUCT SPACES

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Abstract. It is well known in Elementary Geometry the problem proposed and solved by Toricelli in the 17th century, to minimize the sum of the distances of a variable point to three fixed points in the plane.
In this paper, we extend and solve completely the problem in inner product spaces.

1. Introduction

It is well known in Elementary Geometry the problem proposed and solved by Toricelli in the 17th century, to minimize the sum of the distances of a variable point to three fixed points in the plane.

He found that the point for which the minimum is realized is either a vertex of the fixed triangle, if the size of the corresponding angle is greater than $\frac{2\pi}{3}$, or the unique point for each edge is seen under $\frac{2\pi}{3}$.

In this paper we consider and solve Toricelli's problem in a real inner product space H of dimension greater than 1.

Of course, Toricelli's problem can be put in a general metric space (X,d), but if we would like to have nice properties in characterizing Toricellian points, we have to restrict ourselves to the case of normed linear spaces, and in this context, the Hilbertian case should be considered first.

Definition 1. *Let $(X,\|\cdot\|)$ be a real normed space and $\{a_1,a_2,a_3\}$ be a set of three distinct points. We shall say that the element $x_0\in X$ is a Toricellian point for the set $\{a_1,a_2,a_3\}$ provided the following condition holds,*

$$\sum_{i=1}^{3}\|x_0-a_i\|\le\sum_{i=1}^{3}\|x-a_i\| \text{ for all } x\in X. \tag{1.1}$$

That is, the element x_0 minimizes the (nonlinear) functional $T:X\to[0,\infty)$ called Toricelli's functional which is given by

$$T(x):=\sum_{i=1}^{3}\|x-a_i\|.$$

For a given pair of distinct elements $a,b\in X$, $a\neq b$, we define the *right line* determined by the elements a and b by

$$dr(a,b):=\{\lambda a+(1-\lambda)b|\lambda\in\mathbb{R}\}$$

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and the *segment* determined by a and b, by

$$[a,b] := \{\lambda a + (1-\lambda) b | \lambda \in [0,1]\}.$$

A set of points $M \subset X$ will be called *colinear* iff there exists a right line $dr(a,b)$ such that $M \subseteq dr(a,b)$.

Finally, the set of elements $\{a_1, a_2, a_3\} \subset X$ will be called a "triangle" in X if $M_0 := \{a_1, a_2, a_3\}$ is not collinear in the above sense.

Of course, the problem of Toricelli can be extended for $n > 3$ points, but because in that case there is also the possibility of having a number of $m \geq 3$ collinear points which deserve special attention, we restrict ourselves to the original problem of Toricelli. We also note that this simpler case has a valued geometric interpretation as shown by Toricelli himself in the 17th century.

2. The Existence and Unicity

We start with the following lemma which holds in inner product spaces which are not necessarily Hilbert spaces.

Lemma 1. *Let $(H; (\cdot,\cdot))$ be a real inner product space and G a finite-dimensional subspace in H. Then, for all $x \in H$ there exists a unique element $x_1 \in G$ and a unique element $x_2 \in G^{\perp}$ (the orthogonal complement of G) such that*

$$x = x_1 + x_2. \tag{2.1}$$

We denote this by $H = G \bigoplus G^{\perp}$.

Proof. Let $x \in H$. If $x \in G$, then $x = x + 0$ with $0 \in G^{\perp}$ and the decomposition (2.1) holds.

If $x \in X \backslash G$, then, by the well known theorem of best approximation element from finite-dimensional linear subspaces, there exists an element $x_1 \in G$ such that $d(x, x_1) = d(x, G)$.

Put $x_2 := x - x_1$. Then for all $y \in G$ and $\lambda \in \mathbb{R}$, one has

$$\|x_2 + \lambda y\| = \|x - x_1 + \lambda y\| = \|x - (x_1 - \lambda y)\| \geq \|x - x_1\| = \|x_2\|,$$

which is clearly equivalent to $x_2 \perp y$, i.e., $x_2 \in G^{\perp}$ and the representation (2.1) holds.

Now, suppose that there exists another representation $x = y_1 + y_2$ with $y_1 \in G$ and $y_2 \in G^{\perp}$. Then we have

$$G \ni x_1 - y_1 = y_2 - x_2 \in G^{\perp},$$

and since $G \cap G^{\perp} = \{0\}$, we deduce that $x_1 = y_1$ and $x_2 = y_2$ and the unicity in decomposition (2.1) is proved. ∎

The following theorem of existence and unicity for the Toricellian point holds.

Theorem 1. *Let $(H; (\cdot,\cdot))$ be a real inner product space of dimension greater than 1 and $\{a_1, a_2, a_3\}$ a "triangle" in H. Then, there exists a unique Toricellian point associated with this "triangle".*

Proof. The existence. Consider $G = Sp[a_1, a_2, a_3]$ the subspace generated by $\{a_1, a_2, a_3\}$. By Lemma 1, we have $H = G \bigoplus G^{\perp}$.

Now, let $x \in H \backslash G$. Then there exists a unique element $x_1 \in G$ and a unique

$x_2 \in G^{\perp}$ such that $x = x_1 + x_2$ and $x_2 \neq 0$.
For all $a \in G$, we have

$$\begin{aligned} \|x-a\| &= \|x_1 + x_2 - a\| = \|x_2 + (x_1 - a)\| \\ &= \left(\|x_2\|^2 + \|x_1 - a\|^2\right)^{\frac{1}{2}} > \|x_1 - a\| \end{aligned} \tag{2.2}$$

as $\|x_2\| > 0$. Thus, for $a = a_1, a_2, a_3$; we get

$$T(x) = \sum_{i=1}^{3} \|x - a_i\| > \sum_{i=1}^{3} \|x_1 - a_i\| = T(x_1),$$

which shows that the points which minimize the functional T on H are in the finite dimensional space G.
Let $x_0 \in G$. Since $\lim_{\|x\| \to \infty} T(x) = \infty$, there exists a $r > 0$ such that $T(y) > T(x_0)$ for all $y \in G$ with $\|y\| > r$, which shows that the points which minimize T on G are within the closed ball $\bar{B}_g(0, r)$, where

$$\bar{B}_g(0,r) = \bar{B}(0,r) \cap G,$$

and

$$\bar{B}(0,r) := \{x \in H | \, \|x\| \leq y\}$$

is the closed ball in H. As $\bar{B}_g(0,r)$ is compact in the finite-dimensional subspace G and since T is continuous on $\bar{B}_g(0,r)$, it follows, by Weirstrass' Theorem, that there exists an element $x_m \in \bar{B}_g(0,r)$ such that

$$T(x_m) = \inf_{x \in \bar{B}_g(0,r)} T(x);$$

and so x_m is the Toricellian point for the "triangle" $\{a_1, a_2, a_3\}$ in the entire space H.
The unicity. We are going to prove firstly that the Toricellian map $T(x) = \sum_{i=3}^{3} \|x - a_i\|$ is strictly convex on H.
As T is a sum of convex mappings $\|\cdot - a_i\|$, it is obvious that it is a convex map in H and so

$$T(\lambda y_1 + (1-\lambda) y_2) \leq \lambda T(y_1) + (1-\lambda) T(y_2), \tag{2.3}$$

for all $\lambda \in [0,1]$ and $y_1, y_2 \in H$.
Now, let $\lambda \in (0,1)$ and $y_1, y_2 \in H$, with $y_1 \neq y_2$ and assume that the inequality (2.3) becomes an equality, that is,

$$\sum_{i=1}^{3} \|\lambda(y_1 - a_i) + (1-\lambda)(y_2 - a_i)\| = \lambda \sum_{i=1}^{3} \|y_1 - a_i\| + (1-\lambda) \sum_{i=1}^{3} \|y_2 - a_i\|,$$

which implies (by the triangle inequality and an obvious argument) that

$$\|\lambda(y_1 - a_i) + (1-\lambda)(y_2 - a_i)\| = \|\lambda(y_1 - a_i)\| + \|(1-\lambda)(y_2 - a_i)\|, \tag{2.4}$$

for all $i \in \{1,2,3\}$. Taking the square in both parts of (2.4), we get

$$\begin{aligned} & \lambda^2 \|y_1 - a_i\|^2 + 2\lambda(1-\lambda)(y_1 - a_i, y_2 - a_i) + (1-\lambda)^2 \|y_2 - a_i\|^2 \\ = \; & \lambda^2 \|y_1 - a_i\|^2 + 2\lambda(1-\lambda) \|y_1 - a_i\| \|y_2 - a_i\| + (1-\lambda)^2 \|y_2 - a_i\|^2, \end{aligned}$$

which is equivalent to

$$(y_1 - a_i, y_2 - a_i) = \|y_1 - a_i\| \, \|y_2 - a_i\| \text{ for all } i \in \{1, 2, 3\}. \tag{2.5}$$

It is well known that the case of equality in Schwartz's inequality in inner product spaces

$$|(z, u)| \le \|z\| \, \|u\|, \ z, u \in H; \tag{S}$$

holds iff $z = \mu u$, with $\mu \in \mathbb{R}$. Therefore, from (2.5), it follows that there exists $\mu_i \in \mathbb{R}$ $(i = 1, 2, 3)$ such that

$$y_1 - a_i = \mu_i (y_2 - a_i), \ \forall i \in \{1, 2, 3\},$$

which is equivalent to

$$(1 - \mu_i) a_i = y_1 - \mu_i y_2, \ \forall i \in \{1, 2, 3\}. \tag{2.6}$$

Now, if we assume that $\mu_i = 1$, then we get $y_1 = y_2$, which contradicts the above assumption. Hence, $\mu_i \neq 1$ for all $i \in \{1, 2, 3\}$, and then we deduce

$$a_i = \frac{1}{1 - \mu_i} \cdot y_1 - \frac{\mu_i}{1 - \mu_i} \cdot y_2 = \alpha_i y_1 + (1 - \alpha_i) y_2, \ i \in \{1, 2, 3\},$$

where $\alpha_i := \frac{1}{1-\mu_i}$, which contradicts the fact that $\{a_1, a_2, a_3\}$ are not collinear. Consequently, we obtain that T is a strictly convex mapping on H.
Now, suppose that there exist two elements $x_1, x_2 \in H$ with $x_1 \neq x_2$, and so that $T(x_1) = T(x_2) = \inf_{x \in H} T(x) = m$. Consider the convex combination of x_1, x_2, that is, $x_t := t x_1 + (1 - t) x_2$ with $t \in (0, 1)$. Therefore, $x_t \neq x_1, x_2$. Since T is strictly convex, we have

$$T(x_t) = T(t x_1 + (1 - t) x_2) < t T(x_1) + (1 - t) T(x_2) = m,$$

which is a contradiction, and the unicity of the Toricellian point is proved. ∎

3. Characterization of the Toricellian Point

In this section we solve the location problem of the Toricellian point.

Theorem 2. *Let $(H; (\cdot, \cdot))$ be a real inner product space of dimension greater that 1 and $\{a_1, a_2, a_3\}$ a "triangle" in H. The following statements are equivalent*

(i) a_2 is the Toricellian point for $\{a_1, a_2, a_3\}$;

(ii) We have the inequality

$$\left\| \frac{a_1 - a_2}{\|a_1 - a_2\|} + \frac{a_3 - a_2}{\|a_3 - a_2\|} \right\| \le 1; \tag{3.1}$$

(iii) The angle between $a_1 - a_2$ and $a_3 - a_2$ is greater than $\frac{2\pi}{3}$, that is,

$$\cos \theta := \frac{(a_1 - a_2, a_3 - a_2)}{\|a_1 - a_2\| \, \|a_3 - a_2\|} \le -\frac{1}{2}. \tag{3.2}$$

Proof. By the calculation rules in inner product spaces, the condition (3.1) is equivalent to

$$\frac{\|a_1 - a_2\|^2}{\|a_1 - a_2\|^2} + 2 \cdot \frac{(a_1 - a_2, a_3 - a_2)}{\|a_1 - a_2\| \, \|a_3 - a_2\|} + \frac{\|a_3 - a_2\|^2}{\|a_3 - a_2\|^2} \le 1,$$

i.e.,

$$2 + 2 \cdot \frac{(a_1 - a_2, a_3 - a_2)}{\|a_1 - a_2\| \, \|a_3 - a_2\|} \le 1,$$

which is equivalent to (3.2) .

To prove the equivalence "$(i) \Longleftrightarrow (ii)$", we use the following lemma which is a known result in convex optimization.

Lemma 2. *Let $F : X \to \mathbb{R}$ be a convex mapping in the real linear space X. The following statements are equivalent:*

(i) *$x_0 \in X$ minimizes the functional F on X;*

(ii) *One has the inequalities*

$$\lim_{t\to 0+} \frac{F(x_0+tx)-F(x_0)}{t} \geq 0 \geq \lim_{s\to 0-} \frac{F(x_0+sx)-F(x_0)}{s} \tag{3.3}$$

for all $x \in X$.

For the sake of completeness, we give a short proof of this lemma.

Proof. Consider the mapping $\Psi_{x_0,x} : \mathbb{R} \to \mathbb{R}$ given by $\Psi_{x_0,x}(t) := F(x_0+tx)$, $x \in X$.

A simple calculation shows that $\Psi_{x_0,x}$ is convex on $\mathbb{R}$ for all $x \in X$. Hence the following limits exist:

$$\lim_{t\to 0+} \frac{\Psi_{x_0,x}(t)-\Psi_{x_0,x}(0)}{t}, \ \lim_{s\to 0-} \frac{\Psi_{x_0,x}(s)-\Psi_{x_0,x}(0)}{s}$$

and

$$\begin{aligned} \frac{\Psi_{x_0,x}(t)-\Psi_{x_0,x}(0)}{t} &\geq \lim_{t\to 0+} \frac{\Psi_{x_0,x}(t)-\Psi_{x_0,x}(0)}{t} \\ &\geq \lim_{s\to 0-} \frac{\Psi_{x_0,x}(s)-\Psi_{x_0,x}(0)}{s} \\ &\geq \frac{\Psi_{x_0,x}(s)-\Psi_{x_0,x}(0)}{s} \end{aligned}$$

for all $s < 0 < t$. That is, in terms of the mapping F, we have

$$\begin{aligned} \frac{F(x_0+tx)-F(x_0)}{t} &\geq \lim_{t\to 0+} \frac{F(x_0+tx)-F(x_0)}{t} \\ &\geq \lim_{s\to 0-} \frac{F(x_0+sx)-F(x_0)}{s} \\ &\geq \frac{F(x_0+sx)-F(x_0)}{s}, \end{aligned} \tag{3.4}$$

for all $t > 0 > s$.

"$(i) \Rightarrow (ii)$" Assume that x_0 minimizes the functional F. Then, $F(x_0+tx) - F(x_0) \geq 0$ for all $t \in \mathbb{R}$, which implies, by (3.5) , that (3.3) holds.

"$(ii) \Rightarrow (iii)$". If (3.3) holds, then, by (3.4) , we have the inequality

$$\frac{F(x_0+tx)-F(x_0)}{t} \geq 0 \geq \frac{F(x_0+sx)-F(x_0)}{s},$$

for all $t > 0 > s$, which implies that

$$F(x_0+ux) \geq F(x_0)$$

for all $u \in \mathbb{R}$.

Choosing $u = 1$ and $x = y - x_0$, we get $F(y) \geq F(x_0)$ for all $y \in H$, i.e., x_0

minimizes the functional F.

Now, if $T(x) = \sum_{i=1}^{3} \|x - a_i\|$ is the Toricellian functional, then

$$T(x) - T(a_2) = \|x - a_2\| + \sum_{\substack{i=1\\i\neq 2}}^{3} (\|x - a_i\| - \|a_2 - a_i\|)$$
$$= \|x - a_2\| + \sum_{\substack{i=1\\i\neq 2}}^{3} \frac{\|x - a_2\|^2 + 2(x - a_2, a_2 - a_i)}{\|x - a_i\| + \|a_2 - a_i\|}.$$

Let $y \in X$ and $t \in \mathbb{R}$. Then we have

$$T(a_2 + ty) - T(a_2) = |t| \|y\| + \sum_{\substack{i=1\\i\neq 2}}^{3} \frac{t^2 \|y\|^2 + 2t(y, a_2 - a_i)}{\|ty + a_j - a_i\| + \|a_j - a_i\|}.$$

A simple calculation shows that

$$\lim_{t\to 0+} \frac{T(a_2 + ty) - T(a_2)}{t} = \|y\| - \left(y, \sum_{\substack{i=1\\i\neq 2}}^{3} \frac{a_i - a_2}{\|a_i - a_2\|} \right)$$

and

$$\lim_{s\to 0-} \frac{T(a_2 + sy) - T(a_2)}{s} = -\|y\| - \left(y, \sum_{\substack{i=1\\i\neq 2}}^{3} \frac{a_i - a_2}{\|a_i - a_2\|} \right).$$

"$(i) \Rightarrow (ii)$". Assume that a_2 minimizes the functional T. Then, by the implication above, "$(i) \Rightarrow (ii)$" of the above Lemma 2, we have

$$\|y\| - \left(y, \sum_{\substack{i=1\\i\neq 2}}^{3} \frac{a_i - a_2}{\|a_i - a_2\|} \right) \geq 0 \geq -\|y\| - \left(y, \sum_{\substack{i=1\\i\neq 2}}^{3} \frac{a_i - a_2}{\|a_i - a_2\|} \right) \tag{3.5}$$

for all $y \in X$, which is equivalent to

$$\left| \left(y, \sum_{\substack{i=1\\i\neq 2}}^{3} \frac{a_i - a_2}{\|a_i - a_2\|} \right) \right| \leq \|y\| \text{ for all } y \in H. \tag{3.6}$$

Now, if we put in (3.6), $y = u$, where

$$u = \sum_{\substack{i=1\\i\neq 2}}^{3} \frac{a_i - a_2}{\|a_i - a_2\|},$$

we deduce the condition (3.1).

"$(ii) \Rightarrow (i)$". Suppose that (3.1) holds. That is, $\|u\| \leq 1$, where u is as given above. Then, by Schwartz's inequality, we have $|(y, u)| \leq \|y\| \|u\| \leq \|y\|$ for all $y \in H$. That is, (3.6) holds, which is equivalent to (3.5), i.e.,

$$\lim_{t\to 0+} \frac{T(a_2 + ty) - T(a_2)}{t} \geq 0 \geq \lim_{s\to 0-} \frac{T(a_2 + sy) - T(a_2)}{s},$$

which shows that a_2 minimizes the Toricellian functional T. ∎

The second case concerning the location problem of the Toricellian point is embodied in the following theorem.

Theorem 3. *Let $(H;(\cdot,\cdot))$ be a real inner product space of dimension greater than one, $x_0 \in H$ and $\{a_1,a_2,a_3\}$ a "triangle" in H such that none of a_i $\left(i=\overline{1,3}\right)$ are the Toricellian points of $\{a_1,a_2,a_3\}$. Then the following statements are equivalent*

(i) *x_0 is the Toricellian point of $\{a_1,a_2,a_3\}$;*

(ii) *We have $\theta_{12}=\theta_{23}=\theta_{31}=\frac{2\pi}{3}$ where θ_{ij} is the angle between a_i-x_0, a_j-x_0; $(i,j)\in\{(1,2),(2,3),(3,1)\}$, i.e.,*

$$\cos\theta_{ij} := \frac{(a_i-x_0,a_j-x_0)}{\|x_0-a_i\|\,\|x_0-a_j\|} = -\frac{1}{2}, \tag{3.7}$$

for all $(i,j)\in\{(1,2),(2,3),(3,1)\}$.

Proof. Firstly, let us compute

$$\lim_{t\to 0}\frac{T(x+ty)-T(x)}{t},$$

where T is the Toricellian map associated to $\{a_1,a_2,a_3\}$. We have

$$\begin{aligned} &\lim_{t\to 0}\frac{T(x+ty)-T(x)}{t} \\ =\ & \lim_{t\to 0}\sum_{i=1}^{3}\frac{(\|x+ty-a_i\|-\|x-a_i\|)}{t} \\ =\ & \lim_{t\to 0}\left[\frac{1}{t}\sum_{i=1}^{3}\frac{\|x+ty-a_i\|^2-\|x-a_i\|^2}{\|x+ty-a_i\|+\|x-a_i\|}\right] \\ =\ & \lim_{t\to 0}\left[\frac{1}{t}\sum_{i=1}^{3}\frac{2t(y,x-a_i)+t^2\|y\|^2}{\|x+ty-a_i\|+\|x-a_i\|}\right] \\ =\ & \sum_{i=1}^{3}\frac{(y,x-a_i)}{\|x-a_i\|} = \left(y,\sum_{i=1}^{3}\frac{(x-a_i)}{\|x-a_i\|}\right). \end{aligned} \tag{3.8}$$

From Lemma 2, we know that x_0 minimizes the functional T iff

$$\lim_{t\to 0+}\frac{T(x_0+ty)-T(x_0)}{t} \ge 0 \ge \lim_{s\to 0-}\frac{T(x_0+sy)-T(x_0)}{s},\ y\in H, \tag{3.9}$$

and as T is Gatêaux differentiable, we conclude that (3.9) is equivalent to

$$\lim_{t\to 0}\frac{T(x_0+ty)-T(x_0)}{t} = 0, \text{ for all } y\in H, \tag{3.10}$$

that is, by the above equality (3.8),

$$\left(y,\sum_{i=1}^{3}\frac{x_0-a_i}{\|x_0-a_i\|}\right) = 0, \text{ for all } y\in H,$$

which is equivalent to

$$\sum_{i=1}^{3}\frac{x_0-a_i}{\|x_0-a_i\|} = 0. \tag{3.11}$$

Now, if (3.11) holds, then

$$0=\left\langle \sum_{i=1}^{3}\frac{x_0-a_i}{\|x_0-a_i\|},\frac{x_0-a_j}{\|x_0-a_j\|}\right\rangle ,\ \text{for all } j\in\{1,2,3\}\,, \tag{3.12}$$

which is equivalent to:

$$0=\left\langle \sum_{i=1}^{3}\frac{a_i-x_0}{\|a_i-x_0\|},\frac{a_j-x_0}{\|a_j-x_0\|}\right\rangle ,\ \text{for all } j\in\{1,2,3\}\,.$$

Hence, we obtain the system

$$\begin{cases} -1+\cos\theta_{12}+\cos\theta_{13}=0 \\ \cos\theta_{21}-1+\cos\theta_{23}=0 \\ \cos\theta_{31}+\cos\theta_{32}-1=0 \end{cases},$$

and as $\cos\theta_{ij}=\cos\theta_{ji}$, we get $\cos\theta_{ij}=-\frac{1}{2}$.
Conversely, if $\cos\theta_{ij}=-\frac{1}{2}$, then the relation (3.12) holds. If in (3.12) we sum over j from 1 to 3, we obtain

$$0=\sum_{j=1}^{3}\left\langle \sum_{i=1}^{3}\frac{x_0-a_i}{\|x_0-a_i\|},\frac{x_0-a_j}{\|x_0-a_j\|}\right\rangle =\left\|\sum_{i=1}^{3}\frac{x_0-a_i}{\|x_0-a_i\|}\right\|^2,$$

which implies that

$$\sum_{i=1}^{3}\frac{x_0-a_i}{\|x_0-a_i\|}=0,$$

that is,

$$\lim_{t\to 0}\frac{T\left(x_0+ty\right)-T\left(x_0\right)}{t}=0 \text{ for all } y\in H,$$

which is clearly equivalent to the fact that x_0 is the Toricellian point for $\{a_1,a_2,a_3\}$. ∎

Extensions of the above results for n points ($n>3$) in inner product spaces which are not trivial (as we can have a number of $m\geq 3$ points collinearly) will be given in [1].

Some generalizations in normed spaces (Banach spaces, reflexive spaces, strictly convex spaces) are considered in [2].

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