ERROR BOUNDS FOR A GENERAL PERTURBATION OF THE DRAZIN INVERSE

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Abstract. The paper solves a long standing problem of finding error bounds for a general perturbation of the Drazin inverse. The bounds are given in terms of the distance between the matrices together with the distance between their eigenprojections. Estimates using the gap between subspaces are also given. Recent results of several authors, including Castro, Koliha, Straškraba, Wang and Wei can be recovered as special cases of our theorems.

1. Introduction

The necessary and sufficient conditions for the continuity of the Drazin inverse are well known both for matrices [1, 2, 3] and bounded linear operators [6, 8]. However, the quantitative analysis of the perturbation of the Drazin inverse considered in [7, 9] has not made a major progress until recently in the work of Wei and Wang [11], and subsequent papers [4, 5, 10]. The main assumption in [4, 5, 10, 11], in some cases expressed implicitly, is the equality of the eigenprojections of the matrix and its perturbation. In the present paper we offer error bounds for the perturbation of the Drazin inverse under the most general conditions.

For matrix concepts encountered here we refer the reader to the monograph [2] of Campbell and Meyer. In particular, $\mathcal{R}(A)$ will denote the range of $A \in \mathbb{C}^{d\times d}$, $\mathcal{N}(A)$ the nullspace of A, $\sigma(A)$ the spectrum of A, $r(A)$ the spectral radius of A, and A^π the eigenprojection of A corresponding to the eigenvalue 0. The index of A will be written as $i(A)$. Anticipating the future development, we use the following definition of the Drazin inverse, equivalent to the one given in [2].

Definition 1.1. For any matrix $A \in \mathbb{C}^{d\times d}$ we define the *Drazin inverse of* A by

$$A^{\mathrm{D}} = (A + A^\pi)^{-1}(I - A^\pi), \tag{1.1}$$

where A^π is the eigenprojection of A corresponding to 0.

We observe that if A is nonsingular, then $A^\pi = 0$, and $A^{\mathrm{D}} = A^{-1}$. From the definition of the Drazin inverse it follows that

$$\mathcal{R}(A^{\mathrm{D}}) = \mathcal{N}(A^\pi) = \mathcal{R}(A^{i(A)}), \qquad \mathcal{N}(A^{\mathrm{D}}) = \mathcal{R}(A^\pi) = \mathcal{N}(A^{i(A)}). \tag{1.2}$$

In this paper we use exclusively the Euclidean norm $\|x\| = (x^{\mathrm{H}}x)^{1/2}$ for vectors $x \in \mathbb{C}^d$, and the spectral norm

$$\|A\| = r(A^{\mathrm{H}}A)^{1/2}$$

for matrices $A \in \mathbb{C}^{d\times d}$. We recall that

$$\|Ax\| \le \|A\|\|x\| \text{ for all } A \in \mathbb{C}^{d\times d} \text{ and all } x \in \mathbb{C}^d,$$
$$\|AB\| \le \|A\|\|B\| \text{ for all } A,\, B \in \mathbb{C}^{d\times d},$$
$$\|I - P\| = \|P\| \text{ if } P^2 = P \in \mathbb{C}^{d\times d}.$$

For a nonsingular matrix A, $\kappa(A) = \|A\|\|A^{-1}\|$ denotes the condition number of A. As usual, this is generalized to the Drazin condition number $\kappa_{\mathrm{D}}(A) = \|A\|\|A^{\mathrm{D}}\|$ if A is singular.

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2. Error bounds

Given a matrix $A \in \mathbb{C}^{d\times d}$, we want to identify perturbations $B \in \mathbb{C}^{d\times d}$ of A which are close in some sense to A in order to obtain the Drazin inverse B^{D} close to A^{D}, that is, to have $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|$ small. It is clear that $\|B - A\|$ has to be small. Not so obviously, $\|B^\pi - A^\pi\|$ also has to be small; but this is gleaned from the estimate

$$\begin{aligned}\|B^\pi - A^\pi\| &= \|BB^{\mathrm{D}} - AA^{\mathrm{D}}\| = \|(B-A)B^{\mathrm{D}} + A(B^{\mathrm{D}} - A^{\mathrm{D}})\| \\ &\le \|B-A\|\|B^{\mathrm{D}}\| + \|A\|\|B^{\mathrm{D}} - A^{\mathrm{D}}\|.\end{aligned}$$

The error bounds for $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|$ will be derived from the following equation:

$$B^{\mathrm{D}} - A^{\mathrm{D}} = \left[(B+B^\pi)^{-1} - (A+A^\pi)^{-1}\right](I - B^\pi) + (A+A^\pi)^{-1}(A^\pi - B^\pi). \tag{2.1}$$

Suppose that $(\|B-A\| + \|B^\pi - A^\pi\|)\|(A+A^\pi)^{-1}\| < 1$. Applying the standard estimate for the perturbation of the ordinary inverse, we have

$$\begin{aligned}\|(B+B^\pi)^{-1} - (A+A^\pi)^{-1}\| &\le \frac{\|(A+A^\pi)^{-1}\|^2\|(B-A) + (B^\pi - A^\pi)\|}{1 - \|(A+A^\pi)^{-1}\|\|(B-A)+(B^\pi - A^\pi)\|} \\ &\le \frac{\|(A+A^\pi)^{-1}\|^2(\|B-A\| + \|B^\pi - A^\pi\|)}{1 - \|(A+A^\pi)^{-1}\|(\|B-A\| + \|B^\pi - A^\pi\|)}.\end{aligned}$$

Observing that $\|I - B^\pi\| = \|B^\pi\| \le \|A^\pi\| + \|B^\pi - A^\pi\|$, we deduce from (2.1) the following perturbation result:

Theorem 2.1. *Let A, $B \in \mathbb{C}^{d\times d}$ be matrices such that*

$$(\|B-A\| + \|B^\pi - A^\pi\|)\|(A+A^\pi)^{-1}\| < 1. \tag{2.2}$$

Then

$$\begin{aligned}\|B^{\mathrm{D}} - A^{\mathrm{D}}\| \le {} & \frac{\|(A+A^\pi)^{-1}\|^2(\|B-A\| + \|B^\pi - A^\pi\|)}{1 - \|(A+A^\pi)^{-1}\|(\|B-A\| + \|B^\pi - A^\pi\|)}(\|A^\pi\| + \|B^\pi - A^\pi\|) \\ & + \|(A+A^\pi)^{-1}\|\|B^\pi - A^\pi\|.\end{aligned} \tag{2.3}$$

Setting $B^\pi = A^\pi$ in the preceding theorem, we are able to recover most of the perturbation results of [4, 5, 10, 11]. Since the ranks of B^π and A^π are equal if $\|B^\pi - A^\pi\|$ is sufficiently small [2, Proposition 10.7.1], we can also deduce the main result of Campbell and Meyer [3, Theorem 2] on the continuity of the Drazin inverse.

In practice, the quantities $\|(A+A^\pi)^{-1}\|$ and $\|B^\pi - A^\pi\|$ are not easy to estimate directly. In Section 5 we suggest estimates which may be less accurate but easier to evaluate. The preparatory work is done in the next two sections.

3. An estimate for $\|(A+A^\pi)^{-1}\|$

First we note that

$$\begin{aligned}(A+A^\pi)(A^{\mathrm{D}} + A^\pi) &= AA^{\mathrm{D}} + AA^\pi + A^\pi A^{\mathrm{D}} + A^\pi \\ &= I - A^\pi + AA^\pi + A^\pi = I + AA^\pi.\end{aligned}$$

We recall that $(AA^\pi)^{i(A)} = A^{i(A)}A^\pi = 0$, which means that AA^π is nilpotent, and that $I + AA^\pi$ is nonsingular with

$$(I + AA^\pi)^{-1} = \sum_{k=0}^{i(A)-1}(-1)^k(AA^\pi)^k = I + \Big(\sum_{k=1}^{i(A)-1}(-A)^k\Big)A^\pi.$$

Hence

$$\|(A+A^\pi)^{-1}\| = \|(A^{\mathrm{D}} + A^\pi)(I + AA^\pi)^{-1}\| \le \|A^{\mathrm{D}} + A^\pi\|\|(I + AA^\pi)^{-1}\|$$

$$\leq (\|A^{\mathrm{D}}\| + \|A^\pi\|)\Big(1 + \|\sum_{k=1}^{i(A)-1} (-A)^k\|\|A^\pi\|\Big),$$

and

$$\|(A + A^\pi)^{-1}\| \leq (\|A^{\mathrm{D}}\| + \kappa_{\mathrm{D}}(A))\Big(1 + \|\sum_{k=1}^{i(A)-1} (-A)^k\|\kappa_{\mathrm{D}}(A)\Big) =: \Theta, \tag{3.1}$$

when we observe that

$$\|A^\pi\| = \|I - A^\pi\| = \|AA^{\mathrm{D}}\| \leq \|A\|\|A^{\mathrm{D}}\| = \kappa_{\mathrm{D}}(A).$$

4. An estimate for $\|B^\pi - A^\pi\|$

The *gap between subspaces* M, N of $\mathbb{C}^d$ is defined by

$$\operatorname{gap}(M, N) = \max\{\delta(M, N), \delta(N, M)\},$$

where $\delta(M, N) = \sup\{\operatorname{dist}(u, N) : u \in M, \|u\| = 1\}$. For any two matrices A, $B \in \mathbb{C}^{d\times d}$ we define

$$\rho(A, B) = \operatorname{gap}(\mathcal{R}(A^{i(A)}), \mathcal{R}(B^{i(B)})), \quad \nu(A, B) = \operatorname{gap}(\mathcal{N}(A^{i(A)}), \mathcal{N}(B^{i(B)})). \tag{4.1}$$

Recall that A^π is the idempotent matrix with the range $\mathcal{N}(A^{i(A)})$ and the nullspace $\mathcal{R}(A^{i(A)})$. It is known that $\|B^\pi - A^\pi\|$ is small if and only if both $\nu(A, B)$ and $\rho(A, B)$ are small. The following inequalities quantify the 'if' part:

$$\begin{aligned}
\|B^\pi - A^\pi\| &\leq \|(I - A^\pi)B^\pi + A^\pi(I - B^\pi)\| \leq \|(I - A^\pi)B^\pi\| + \|A^\pi(I - B^\pi)\| \\
&\leq \|I - A^\pi\|\|B^\pi\|\nu(A, B) + \|A^\pi\|\|I - B^\pi\|\rho(A, B) \\
&= \|A^\pi\|\|B^\pi\|(\nu(A, B) + \rho(A, B)) \\
&\leq \|A^\pi\|(\|A^\pi\| + \|B^\pi - A^\pi\|)(\nu(A, B) + \rho(A, B)) \\
&\leq \kappa_{\mathrm{D}}(A)(\kappa_{\mathrm{D}}(A) + \|B^\pi - A^\pi\|)(\nu(A, B) + \rho(A, B)),
\end{aligned}$$

which implies $(1 - \kappa_{\mathrm{D}}(A)(\nu(A, B) + \rho(A, B)))\,\|B^\pi - A^\pi\| \leq \kappa_{\mathrm{D}}^2(A)(\nu(A, B) + \rho(A, B))$. Suppose that

$$\kappa_{\mathrm{D}}(A)(\nu(A, B) + \rho(A, B))) < 1. \tag{4.2}$$

Then

$$\|B^\pi - A^\pi\| \leq \frac{\kappa_{\mathrm{D}}^2(A)(\nu(A, B) + \rho(A, B))}{1 - \kappa_{\mathrm{D}}(A)(\nu(A, B) + \rho(A, B))} =: \Delta. \tag{4.3}$$

5. The final estimate

We observe that the factor in the form of a fraction on the right in (2.3) can be written as $\varphi(t) = t/(1 - t)$ with $0 \leq t < 1$, where φ is increasing in t. Therefore in (2.3) we can replace $\|(A + A^\pi)^{-1}\|$ by Θ, and $\|B^\pi - A^\pi\|$ by Δ, provided the relevant inequalities for Θ and Δ are satisfied. We then have the following result.

Theorem 5.1. *Let A, $B \in \mathbb{C}^{d\times d}$ be matrices with $\nu(A, B)$, $\rho(A, B)$, Θ and Δ defined by* (4.1), (3.1) *and* (4.3), *respectively, such that* (4.2) *holds, and that*

$$\Theta(\|B - A\| + \Delta) < 1. \tag{5.1}$$

Then

$$\|B^{\mathrm{D}} - A^{\mathrm{D}}\| \leq \frac{\Theta^2(\|B - A\| + \Delta)}{1 - \Theta(\|B - A\| + \Delta)}(\kappa_{\mathrm{D}}(A) + \Delta) + \Theta\Delta. \tag{5.2}$$

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