OTHER PROOFS OF MONOTONICITY FOR GENERALIZED WEIGHTED MEAN VALUES

FENG QI, JIA-QIANG MEI, AND SEN-LIN XU

Abstract. In this article, another two simple and short proofs of monotonicity for the generalized weighted mean values with two parameters are given.

1. Introduction

The generalized weighted mean values $M_{p,f}(r,s;x,y)$ with two parameters r and s are defined by the first author in [5] as follows:

Let $x,y,r,s\in\mathbb{R}$, $p(u)\not\equiv 0$ be a nonnegative and integrable function and $f(u)$ a positive and integrable function on the interval between x and y, then

$$(1.1)\qquad M_{p,f}(r,s;x,y) = \left(\frac{\int_x^y p(u)f^s(u)\,du}{\int_x^y p(u)f^r(u)\,du}\right)^{\frac{1}{s-r}},\ (r-s)(x-y)\neq 0;$$

$$(1.2)\qquad M_{p,f}(r,r;x,y) = \exp\left(\frac{\int_x^y p(u)f^r(u)\ln f(u)\,du}{\int_x^y p(u)f^r(u)\,du}\right),\ x-y\neq 0;$$

$$M_{p,f}(r,s;x,x) = f(x).$$

For our own convenience, we write

$$M_{p,f}(r,s;x,y)=M_{p,f}(r,s)=M_{p,f}(x,y)=M_{p,f},$$

shifting notations to suit the context.

Note that most two variable mean values are special cases of $M_{p,f}$. If $s=0$, then $M_{p,f}(r,0;x,y)=M^{[r]}(f,p;x,y)$ is called the weighted mean of order r of the function f on the interval between x and y with weight p in [3] and [4]. If we take $p(u)\equiv 1$, $f(u)=u$ and $x,y>0$, then $M_{p,f}(r-1,s-1;x,y)=E(r,s;x,y)$ are called the extended mean values in [1] and [4].

The extended mean values E are increasing with r and s, or with x and y. It has been proven by many mathematicians, for instance [1], [2], [4], [7], [9] and [12]. The study of E has a rich literature, for details, please see [5]. The monotonicity of $M_{p,f}$ was verified by the first author in [5] and [8] using the Chebychev integral inequality, the Cauchy-Schwarz-Buniakowsky integral inequality, and the mean value theorem.

In this article, from the ideas and viewpoints used in [6], [9], [10] and [11], we prove the monotonicity of $M_{p,f}(r,s;x,y)$ by two new and simple methods. That is

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Theorem 1. *Let $p(u) \not\equiv 0$ be a nonnegative and continuous function, $f(u)$ a positive, increasing (or decreasing, respectively) and continuous function. Then the generalized weighted mean values $M_{p,f}(r,s;x,y)$ increase (or decrease, respectively) with respect to either x or y,*

2. The First Proof of the Theorem

Let

$$h_{p,f}(t;x,y) = \int_x^y p(u) f^t(u)\, du,\ t \in \mathbb{R}, \tag{2.1}$$

where x, y, p and f are defined as stated in Section 1.

It is easy to see that

$$\frac{\partial^n h_{p,f}(t;x,y)}{\partial t^n} = \int_x^y p(u) f^t(u) [\ln f(u)]^n\, du. \tag{2.2}$$

Set $Q_{p,f}(r,s;x,y) = \ln M_{p,f}(r,s;x,y)$, then

$$Q_{p,f}(r,s;x,y) = \frac{1}{s-r}\int_r^s \frac{\frac{\partial h_{p,f}(t;x,y)}{\partial t}}{h_{p,f}(t;x,y)} dt,\ (r-s)(x-y) \neq 0; \tag{2.3}$$

$$Q_{p,f}(r,r;x,y) = \frac{\frac{\partial h_{p,f}(r;x,y)}{\partial r}}{h_{p,f}(r;x,y)},\ x,y \neq 0. \tag{2.4}$$

To verify the monotonicity of $M_{p,f}(r,s;x,y)$ with x and y, it is sufficient to prove the monotonicity of $\frac{\frac{\partial h_{p,f}(t;x,y)}{\partial t}}{h_{p,f}(r;x,y)}$ in $Q_{p,f}(r,s;x,y)$ with x and y for any t. This is a special case of the following

Lemma 1. *The functions*

$$\frac{\frac{\partial^{2(k+i)+1} h_{p,f}(t;x,y)}{\partial t^{2(k+i)+1}}}{\frac{\partial^{2k} h_{p,f}(t;x,y)}{\partial t^{2k}}} \tag{2.5}$$

are increasing (or decreasing, respectively) with x and y if $f(u)$ is increasing (or decreasing, respectively) for i and k being nonnegative integers.

Proof. Using the integral expressions (2.1) and (2.2) of $h_{p,f}(t;x,y)$, by standard arguments, we have

$$\begin{aligned}
&\frac{\partial}{\partial y}\left(\frac{\frac{\partial^{2(k+i)+1} h_{p,f}(t;x,y)}{\partial t^{2(k+i)+1}}}{\frac{\partial^{2k} h_{p,f}(t;x,y)}{\partial t^{2k}}}\right)\\
&= \left[\frac{\partial}{\partial y}\left(\frac{\partial^{2(k+i)+1} h_{p,f}(t;x,y)}{\partial t^{2(k+i)+1}}\right)\cdot \frac{\partial^{2k} h_{p,f}(t;x,y)}{\partial t^{2k}}\right.\\
&\quad \left. - \frac{\partial^{2(k+i)+1} h_{p,f}(t;x,y)}{\partial t^{2(k+i)+1}} \cdot \frac{\partial}{\partial y}\left(\frac{\partial^{2k} h_{p,f}(t;x,y)}{\partial t^{2k}}\right)\right] \cdot \frac{1}{\left[\frac{\partial^{2k} h_{p,f}(t;x,y)}{\partial t^{2k}}\right]^2}\\
&= \frac{p(y) f^t(y) [\ln f(y)]^{2k}}{\left[\frac{\partial^{2k} h_{p,f}(t;x,y)}{\partial t^{2k}}\right]^2} \cdot \left[(\ln f(y))^{2i+1} \int_x^y p(u) f^t(u) [\ln f(u)]^{2k}\, du\right.\\
&\quad \left. - \int_x^y p(u) f^t(u) [\ln f(u)]^{2(i+k)+1}\, du\right].
\end{aligned} \tag{2.6}$$

When $f(u)$ increases (or decreases, respectively), the derivatives (2.6) are nonnegative (or nonpositive, respectively); hence, the desired monotonicity of (2.5) with respect to x and y follows, since the discussed functions (2.5) are symmetric in x and y. This completes the proof of the lemma. ∎

3. The Second Proof of the Theorem

Let

$$\alpha(t) = \frac{p(y) f^t(y)}{\int_x^y p(u) f^t(u)\,du}. \tag{3.1}$$

Straightforward computation yields

$$\alpha'(t) = \frac{p(y) f^t(y) \int_x^y p(u) f^t(u) \ln \frac{f(y)}{f(u)} du}{\left(\int_x^y p(u) f^t(u)\,du\right)^2} \geq 0. \tag{3.2}$$

By straightforward computation, from the mean-value theorem, we know that there is at least one point ξ between r and s such that

$$\frac{\frac{\partial M_{p,f}(r,s;x,y)}{\partial y}}{M_{p,f}(r,s;x,y)} = \frac{\alpha(s) - \alpha(r)}{s-r} = \alpha'(\xi) \geq 0, \tag{3.3}$$

thus, we obtain that the generalized weighted mean values $M_{p,f}(r,s;x,y)$ increase in y and x, since $M_{p,f}(r,s;x,y)$ is symmetric with x and y.

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Department of Mathematics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, The People's Republic of China
E-mail address: qifeng@jzit.edu.cn

(Jia-Qiang Mei & Sen-Lin Xu) Department of Mathematics, University of Science and Technology of China, Hefei City, Anhui 230026, The People's Republic of China