INEQUALITIES AND MONOTONICITY OF SEQUENCES INVOLVING $\sqrt[n]{(n+k)!/k!}$

FENG QI

ABSTRACT. Using Stirling's formula, for all nonnegative integers k and natural numbers n and m, we prove that

$$\left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \Bigg/ \left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)} \le \sqrt{\frac{n+k}{n+m+k}}.$$

From this, some monotonicity results of sequences involving $\sqrt[n]{(n+k)!/k!}$ are deduced, and the related inequalities are refined.

1. INTRODUCTION

In [4], H. Minc and L. Sathre proved that, if r is a positive integer and $\phi(r) = (r!)^{1/r}$, then

$$1 < \phi(r+1)/\phi(r) < (r+1)/r. \tag{1}$$

In [1, 3], H. Alzer and J. S. Martins refined the right inequality in (1) and showed that, if n is a positive integer, then we have for all positive real numbers r,

$$\frac{n}{n+1} < \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Bigg/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{2}$$

Both bounds in (2) are the best possible.

Let n and m be natural numbers, k a nonnegative integer. The author generalized in [8] the left inequality in (2) and obtained

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r \Bigg/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r}, \tag{3}$$

where r is a given positive real number. The lower bound in (3) is the best possible.

By mathematical induction, N. Elezović and J. Pečarić [2] and the author [6, 7] further generalized the left side of inequalities in (2) and the inequality (3) to a large class of positive, increasing and convex sequences and of positive, increasing and logarithmically concave sequences in different directions, respectively.

Recently, the inequalities in (1) were generalized by the author and Q.-M. Luo in [10] and the following inequalities were obtained:

$$\frac{n+k}{n+m+k} < \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}} < 1, \tag{4}$$

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 26A48.
Key words and phrases. Inequality, monotonicity, Stirling's formula, sequence.
The author was supported in part by NSF of Henan Province, The People's Republic of China.
This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX.

where n and m are natural numbers, k is a nonnegative integer. Meanwhile, some monotonicity results of the sequences involving $\sqrt[n]{(n+k)!/k!}$ were presented.

In this article, using inequalities deduced from Stirling's formula, we give a refinement of the left inequality in (1) and the right inequality in (4), that is

Theorem 1. *Let k be a nonnegative integer, n and m be natural numbers, then*

$$\left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \Bigg/ \left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)} \leq \sqrt{\frac{n+k}{n+m+k}}. \tag{5}$$

When $n=m=1$, the equality in (5) *is valid.*

Theorem 2. *The sequences*

$$\left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \Bigg/ \sqrt{n+k} \tag{6}$$

are strictly increasing with n and k, respectively. The sequences

$$\left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \Bigg/ \left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)} \tag{7}$$

are strictly increasing in k for all given natural numbers n and m.

2. Lemmas

In order to verify our theorems, we need some lemmas.

Lemma 1. *For all natural numbers n, we have*

$$n\ln n - n \leq \ln n! \leq \left(n+\frac{1}{2}\right)\ln n - n + 1, \tag{8}$$

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\frac{1}{12n+1} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\frac{1}{12n}. \tag{9}$$

For details about inequalities (8) and (9), please refer to [5, p. 184 and p. 194].

Lemma 2. *For all natural numbers k and $n>1$, we have*

$$\frac{n(n-1)}{2}\ln\left(1+\frac{1}{n+k}\right) + \frac{2k+1}{2}\ln\left(1+\frac{n}{k}\right) \leq n + \frac{12n-1}{12(n+k)(12k+1)}. \tag{10}$$

Proof. Let

$$\varphi(x,y) = \frac{x(x-1)}{2}\ln\left(1+\frac{1}{x+y}\right) + \frac{2y+1}{2}\ln\left(1+\frac{x}{y}\right) - \frac{12x-1}{12(x+y)(12y+1)} - x, \quad x\geq 2, \quad y \geq 1.$$

Differentiating and simplifying produces

$$\frac{\partial\varphi(x,y)}{\partial x} = \frac{2x-1}{2}\ln\left(1+\frac{1}{x+y}\right) + \frac{6y^2+5y-1+(5+6y-12y^2)x-30yx^2-18x^3}{12(x+y)^2(x+y+1)}.$$

Using the inequality $\ln(1+t) \leq \frac{t(2+t)}{2(1+t)}$ for $t\geq 0$ in [9], we have

$$\frac{\partial\varphi(x,y)}{\partial x} \leq \frac{2y-1+2x-6yx^2-6x^3}{12(x+y)^2(x+y+1)} \leq 0,$$

therefore $\varphi(x,y)$ is decreasing with $x \geq 2$.

It is easy to see that

$$\varphi(2,y)=\ln\left(1+\frac{1}{y+2}\right)+\frac{2y+1}{2}\ln\left(1+\frac{2}{y}\right)-\frac{23}{12(y+2)(12y+1)}-2,$$
$$\varphi(2,1)=2\ln 2+\frac{1}{2}\ln 3-\frac{23}{468}-2<0.$$

Denote

$$\psi(y)=\varphi(2,y)+\frac{23}{12(y+2)(12y+1)}, \tag{11}$$

then

$$\psi'(y)=\ln\left(1+\frac{2}{y}\right)-\frac{2y^2+8y+3}{y(y+2)(y+3)},$$
$$\psi''(y)=\frac{18-6y-5y^2}{[y(y+2)(y+3)]^2}.$$

Thus $\psi'(y)$ is decreasing with $y\geq 2$. Since $\lim_{y\to+\infty}\psi'(y)=0$, then $\psi'(y)\geq 0$ for $y\geq 2$ and $\psi(y)$ increases for $y\geq 2$.

Also since $\lim_{y\to+\infty}\psi(y)=0$, we get $\psi(y)<0$ for $y\geq 2$ and $\varphi(2,y)<0$ for $y\geq 2$. Hence the sequence $\{\varphi(2,k)\}_{k=1}^{+\infty}$ is negative, and so is the sequence $\{\varphi(n,k)\}_{n=2,k=1}^{+\infty}$. Lemma 2 follows. ∎

3. Proofs of Theorems

Proposition 1. *Let n be a natural number, then*

$$\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}\leq\sqrt{\frac{n}{n+1}}. \tag{12}$$

When $n=1$, the equality in (12) holds.

Proof. Let

$$f(x)=x(x-1)\ln(1+x)-(x^2-x-1)\ln x-2x+2,\quad x\geq 1.$$

By standard arguments, we get

$$\begin{aligned}f'(x)&=(2x-1)\ln\left(1+\frac{1}{x}\right)+\frac{x(x-1)}{1+x}-\frac{x^2-x-1}{x}-2\\&\leq\frac{2x-1}{x}+\frac{x(x-1)}{1+x}-\frac{x^2-x-1}{x}-2\\&=\frac{1-x}{1+x}\leq 0,\\f(1)&=0.\end{aligned}$$

Therefore, $f(x)$ is decreasing and $f(x)\leq 0$ for $x\geq 1$. And for $n\geq 1$, we obtain

$$n(n-1)\ln(1+n)+2\leq(n^2-n-1)\ln n+2n,$$

this is equivalent to

$$\left(n+\frac{1}{2}\right)\ln n-n+1\leq\frac{n(n+1)}{2}\ln n-\frac{n(n-1)}{2}\ln(1+n).$$

Thus, from the right-hand side of inequality (8), we obtain

$$\ln n! \le \frac{n(n+1)}{2}\ln n - \frac{n(n-1)}{2}\ln(1+n),$$

that is,

$$\begin{aligned} n! &\le \frac{n^{n(n+1)/2}}{(n+1)^{n(n-1)/2}}, \\ \frac{n!}{(n+1)^n} &\le \frac{n^{n(n+1)/2}}{(n+1)^{n(n+1)/2}}, \\ \sqrt{\frac{n}{n+1}} &\ge \left(\frac{(n!)^{n+1}}{[(n+1)!]^n}\right)^{1/n(n+1)}. \end{aligned}$$

The proof is complete. ∎

Proposition 2. *Let k and n be natural numbers, then*

$$\left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \Bigg/ \left(\prod_{i=k+1}^{n+k+1} i\right)^{1/(n+1)} \le \sqrt{\frac{n+k}{n+k+1}}. \tag{13}$$

If $n=1$, the equality in (13) is valid for all $k \ge 1$.

Proof. For $n>1$, the inequality (10) may be rewritten as

$$\frac{n(n-1)}{2}\ln\frac{n+k+1}{n+k} + \frac{2k+1}{2}\ln\frac{n+k}{k} \le n + \frac{12n-1}{12(n+k)(12k+1)},$$

$$\frac{n(n-1)}{2}\ln(n+k+1) + \frac{n-n^2+2k+1}{2}\ln(n+k)$$
$$\le \left(k+\frac{1}{2}\right)\ln k + n + \frac{12n-1}{12(n+k)(12k+1)},$$

$$\left(n+k+\frac{1}{2}\right)\ln(n+k) - \left(k+\frac{1}{2}\right)\ln k - n + \frac{1}{12(n+k)} - \frac{1}{12k+1}$$
$$\le \frac{n(n+1)}{2}\ln(n+k) - \frac{n(n-1)}{2}\ln(n+k+1).$$

Then, substituting inequalities in (9) into the final inequality above yields

$$\ln(n+k)! - \ln k! \le \frac{n(n+1)}{2}\ln(n+k) - \frac{n(n-1)}{2}\ln(n+k+1),$$

this inequality can be rearranged as

$$\begin{aligned} \frac{(n+k)!}{k!} &\le \frac{(n+k)^{n(n+1)/2}}{(n+k+1)^{n(n-1)/2}}, \\ \prod_{i=k+1}^{n+k} i \Bigg/ (n+k+1)^n &\le \left(\frac{n+k}{n+k+1}\right)^{n(n+1)/2}, \\ \sqrt{\frac{n+k}{n+k+1}} &\ge \left[\left(\prod_{i=k+1}^{n+k} i\right)^n \left(\prod_{i=k+1}^{n+k} i\right) \Bigg/ \left(\prod_{i=k+1}^{n+k+1} i\right)^n\right]^{1/n(n+1)}. \end{aligned}$$

From this, the inequality (13) follows. ∎

Proofs of Theorem 1 and Theorem 2. The inequality (5) can be rewritten as

$$\frac{\left(\prod_{i=k+1}^{n+k} i\right)^{1/n}}{\sqrt{n+k}} \le \frac{\left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)}}{\sqrt{n+m+k}},$$

which is equivalent to

$$\frac{\left(\prod_{i=k+1}^{n+k} i\right)^{1/n}}{\sqrt{n+k}} \le \frac{\left(\prod_{i=k+1}^{n+k+1} i\right)^{1/(n+1)}}{\sqrt{n+k+1}},$$

that is,

$$\left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \Bigg/ \left(\prod_{i=k+1}^{n+k+1} i\right)^{1/(n+1)} \le \sqrt{\frac{n+k}{n+k+1}}.$$

This follows from the combination of Proposition 1 and Proposition 2. Thus, Theorem 1 and the monotonicity of the sequences $\left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \Big/ \sqrt{n+k}$ being increasing with n are proved.

It is easy to see that

$$\frac{\left(\prod_{i=k+2}^{n+k+1} i\right)^{1/n}}{\sqrt{n+k+1}} = \sqrt{\frac{n+k}{n+k+1}}\left(\frac{n+k+1}{k+1}\right)^{1/n} \frac{\left(\prod_{i=k+1}^{n+k} i\right)^{1/n}}{\sqrt{n+k}}.$$

Let

$$g(x) = \left(\frac{1}{x} - \frac{1}{2}\right)\ln(x+k+1) - \frac{1}{x}\ln(k+1) + \frac{1}{2}\ln(x+k), \quad x \ge 1.$$

Using $\frac{2t}{2+t} \le \ln(1+t)$ for $t > 0$ in [5, pp. 273–274] and [9] and differentiating yields

$$\begin{aligned} g'(x) &= \frac{1}{x^2}\left[\frac{x(3x+2k)}{2(x+k)(x+k+1)} + \ln\frac{k+1}{x+k+1}\right] \\ &\le \frac{1}{x^2}\left[\frac{x(3x+2k)}{2(x+k)(x+k+1)} - \frac{2x}{x+2(k+1)}\right] \\ &= \frac{2-x}{2(x+k)(x+k+1)[x+2(k+1)]}. \end{aligned}$$

Thus, $g(x)$ is decreasing with $x \ge 2$. Since $\lim_{x\to+\infty} g(x) = 0$ and $g(1) = \frac{1}{2}\ln\frac{k+2}{k+1} > 0$, then the sequence $\{g(n)\}_{n=1}^{+\infty}$ is positive. Hence

$$\sqrt{\frac{n+k}{n+k+1}}\left(\frac{n+k+1}{k+1}\right)^{1/n} > 1,$$

$$\frac{\left(\prod_{i=k+2}^{n+k+1} i\right)^{1/n}}{\sqrt{n+k+1}} > \frac{\left(\prod_{i=k+1}^{n+k} i\right)^{1/n}}{\sqrt{n+k}}.$$

The sequences $\left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \Big/ \sqrt{n+k}$ being increasing with k are verified.

It is clear that

$$\begin{aligned}
&\frac{\sqrt[n]{(n+k+1)!/(k+1)!}}{\sqrt[n+m]{(n+m+k+1)!/(k+1)!}}\\
={}&\left[\frac{(n+k+1)^{n+m}}{(k+1)^m(n+m+k+1)^n}\right]^{1/n(n+m)}\frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}}.
\end{aligned}$$

Let

$$h(x)=(x+m)\ln(x+k+1)-x\ln(x+m+k+1)-m\ln(k+1),\quad x\geq 0.$$

Direct calculation leads to

$$\begin{aligned}
h'(x)&=\frac{m(2x+m+k+1)}{(x+k+1)(x+m+k+1)}-\ln\left(1+\frac{m}{x+k+1}\right)\\
&\geq\frac{m(2x+m+k+1)}{(x+k+1)(x+m+k+1)}-\frac{m}{x+k+1}\\
&=\frac{mx}{(x+k+1)(x+m+k+1)}\geq 0,\\
h(0)&=0.
\end{aligned}$$

So, the function $h(x)\geq 0$ for $x\geq 0$, and we have

$$\frac{(n+k+1)^{n+m}}{(k+1)^m(n+m+k+1)^n}>1,$$

$$\left(\prod_{i=k+2}^{n+k+1}\right)^{1/n}\Bigg/\left(\prod_{i=k+2}^{n+m+k+1}\right)^{1/(n+m)}>\left(\prod_{i=k+1}^{n+k}i\right)^{1/n}\Bigg/\left(\prod_{i=k+1}^{n+m+k}i\right)^{1/(n+m)}.$$

The proof of Theorem 2 is complete.

References

[1] H. Alzer, *On an inequality of H. Minc and L. Sathre*, J. Math. Anal. Appl. **179** (1993), 396–402.

[2] N. Elezović and J. Pečarić, *On Alzer's inequality*,J. Math. Anal. Appl. **223** (1998), 366–369.

[3] J. S. Martins, *Arithmetic and geometric means, an applications to Lorentz sequence spaces*, Math Nachr. **139** (1988), 281–288.

[4] H. Minc and L. Sathre, *Some inequalities involving* $(r!)^{1/r}$, Proc. Edinburgh Math. Soc. **14** (1964/65), 41–46.

[5] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, New York/Heidelberg/Berlin, 1970.

[6] Feng Qi, *An algebraic inequality*, RGMIA Research Report Collection **2** (1999), no. 1, 71–72.

[7] Feng Qi, *Further generalization of H. Alzer's inequality*,Intern. J. Math. & Math. Sci. **25** (2000), in the press.

[8] Feng Qi, *Generalization of H. Alzer's inequality*, J. Math. Anal. Appl. (1999), in the press.

[9] Feng Qi and Chao-Ping Chen, *Monotonicities of two sequences*, Mathematics and Informatics Quarterly (1999), in the press.

[10] Feng Qi and Qiu-Ming Luo, *Generalization of H. Minc and J. Sathre's inequality*, Tamkang Journal of Mathematics **31** (2000), no. 2, in the press.

[11] Feng Qi, *Generalizations of Alzer's and Kuang's inequality*, submitted for publication.

Department of Mathematics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, The People's Republic of China

E-mail address: qifeng@jzit.edu.cn