LOGARITHMIC CONVEXITIES OF THE EXTENDED MEAN VALUES

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ABSTRACT. In this article, the logarithmic convexities of the extended mean values are proved, an inequality of mean values is presented. As by-products, two analytic inequalities are derived, and two open problems are proposed.

1. Introduction

The so-called extended mean values $E(r, s; x, y)$ were defined first by Stolarsky in [17] as

$$E(r,s;x,y) = \left[\frac{r}{s}\cdot\frac{y^s-x^s}{y^r-x^r}\right]^{1/(s-r)}, \qquad rs(r-s)(x-y)\neq 0; \tag{1.1}$$

$$E(r,0;x,y) = \left[\frac{1}{r}\cdot\frac{y^r-x^r}{\ln y-\ln x}\right]^{1/r}, \qquad r(x-y)\neq 0; \tag{1.2}$$

$$E(r,r;x,y) = \mathrm{e}^{-1/r}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)}, \qquad r(x-y)\neq 0; \tag{1.3}$$

$$E(0,0;x,y) = \sqrt{xy}, \qquad x\neq y; \tag{1.4}$$

$$E(r,s;x,x) = x, \qquad x = y.$$

Define a function g by

$$g(t) = g(t;x,y) = \begin{cases} \dfrac{(y^t-x^t)}{t}, & t\neq 0;\\ \ln y-\ln x, & t=0. \end{cases} \tag{1.5}$$

It is easy to see that g can be expressed in integral form as

$$g(t;x,y) = \int_x^y u^{t-1}du, \quad t\in\mathbb{R}, \tag{1.6}$$

and

$$g^{(n)}(t) = \int_x^y (\ln u)^n u^{t-1}du, \quad t\in\mathbb{R}. \tag{1.7}$$

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Therefore, the extended mean values can be represented [7, 15] in terms of g by

$$E(r,s;x,y)=\begin{cases}\left(\dfrac{g(s;x,y)}{g(r;x,y)}\right)^{1/(s-r)}, & (r-s)(x-y)\neq 0;\\ \exp\left(\dfrac{\partial g(r;x,y)/\partial r}{g(r;x,y)}\right), & r=s, \quad x-y\neq 0\end{cases} \tag{1.8}$$

and

$$\ln E(r,s;x,y)=\begin{cases}\dfrac{1}{s-r}\displaystyle\int_r^s\dfrac{\partial g(t;x,y)/\partial t}{g(t;x,y)}\,dt, & (r-s)(x-y)\neq 0;\\ \dfrac{\partial g(r;x,y)/\partial r}{g(r;x,y)}, & r=s, \quad x-y\neq 0.\end{cases} \tag{1.9}$$

Leach and Sholander [2] showed that $E(r,s;x,y)$ are increasing with both r and s, or with both x and y. The monotonicities of E have also been researched by the author and others in [11]– [15] using different ideas and simpler methods.

Leach and Sholander [3] and Páles [6] solved the problem of comparison of E; that is, they found necessary and sufficient conditions for the parameters r,s,u,v in order that

$$E(r,s;x,y)\leq E(u,v;x,y) \tag{1.10}$$

be satisfied for all positive x and y.

Most of two variable means are special cases of E, for example [1],

$$E(1,2;x,y)=A(x,y), \quad E(1,1;x,y)=I(x,y), \quad E(0,1;x,y)=L(x,y). \tag{1.11}$$

They are called the arithmetic mean, the identric mean, and the logarithmic mean, respectively.

The main purpose of this paper is to verify the logarithmic convexities of the extended mean values $E(r,s;x,y)$. As applications, one inequality among the arithmetic mean, the identric mean and the logarithmic mean is established; two open problems are proposed. As by-products, an inequality for the exponential function is obtained.

2. Logarithmic Convexities of $E(r,s;x,y)$

In order to prove our main result, the following lemma is necessary.

Lemma ([15])**.** *Assume that the derivative of second order $f''(t)$ exists on $\mathbb{R}$. If $f(t)$ is an increasing (or convex, respectively) function on $\mathbb{R}$, then the arithmetic mean of function $f(t)$,*

$$\phi(r,s)=\begin{cases}\dfrac{1}{s-r}\displaystyle\int_r^s f(t)\,dt, & r\neq s,\\ f(r), & r=s,\end{cases} \tag{2.1}$$

is also increasing (or convex, respectively) with both r and s on $\mathbb{R}$.

Proof. Direct calculation yields

$$\frac{\partial \phi(r,s)}{\partial s} = \frac{1}{(s-r)^2}\Big[(s-r)f(s) - \int_r^s f(t)dt\Big], \tag{2.2}$$

$$\frac{\partial^2 \phi(r,s)}{\partial s^2} = \frac{(s-r)^2 f'(s) - 2(s-r)f(s) + 2\int_r^s f(t)dt}{(s-r)^3} \equiv \frac{\varphi(r,s)}{(s-r)^3}, \tag{2.3}$$

$$\frac{\partial \varphi(r,s)}{\partial s} = (s-r)^2 f''(s). \tag{2.4}$$

In the case of $f'(t) \geq 0$, we have $\partial\phi(r,s)/\partial s \geq 0$, thus $\phi(r,s)$ increases in both r and s, since $\phi(r,s) = \phi(s,r)$.

In the case of $f''(t) \geq 0$, $\varphi(r,s)$ increases with s. Since $\varphi(r,r) = 0$, we have $\partial^2\phi(r,s)/\partial s^2 \geq 0$. Therefore $\phi(r,s)$ is convex with respect to either r or s, since $\phi(r,s) = \phi(s,r)$. This completes the proof. ∎

By formula (1.9) and the above Lemma, in order to prove the logarithmic convexities of the extended mean values $E(r,s;x,y)$, it suffices to verify the convexities of the function $g'(t)/g(t) \triangleq g'_t(t;x,y)/g(t;x,y) \triangleq \big(\partial g(t;x,y)/\partial t\big)/g(t;x,y)$ with respect to t, where $g(t) = g(t;x,y)$ is defined by (1.5) or (1.6).

Straightforward computation results in

$$\left(\frac{g'(t)}{g(t)}\right)' = \frac{g''(t)g(t) - [g'(t)]^2}{g^2(t)}, \tag{2.5}$$

$$\left(\frac{g'(t)}{g(t)}\right)'' = \frac{g^2(t)g'''(t) - 3g(t)g'(t)g''(t) + 2[g'(t)]^3}{g^3(t)}. \tag{2.6}$$

For $y > x = 1$, expanding $g(t;1,y)$ into series at $t_0 = 0$ with respect to t directly gives us

$$\begin{aligned} g(t;1,y) &= \sum_{i=0}^{\infty} \frac{(\ln y)^{i+1}}{(i+1)!} t^i, \quad g'_t(t;1,y) = \sum_{i=0}^{\infty} \frac{(i+1)(\ln y)^{i+2}}{(i+2)!} t^i, \\ g''_t(t;1,y) &= \sum_{i=0}^{\infty} \frac{(i+1)(i+2)(\ln y)^{i+3}}{(i+3)!} t^i, \\ g'''_t(t;1,y) &= \sum_{i=0}^{\infty} \frac{(i+1)(i+2)(i+3)(\ln y)^{i+4}}{(i+4)!} t^i. \end{aligned} \tag{2.7}$$

From the four fundamental operations of arithmetic and suitable properties of series, we have

$$g^2(t;1,y) = 2\sum_{k=0}^{\infty} \frac{(2^{k+1}-1)(\ln y)^{k+2}}{(k+2)!} t^k, \tag{2.8}$$

$$g(t;1,y)g'_t(t;1,y) = \sum_{k=0}^{\infty} \frac{(k+1)(2^{k+2}-1)(\ln y)^{k+3}}{(k+3)!} t^k, \tag{2.9}$$

(2.10)

$$g(t;1,y)g''_t(t;1,y) + [g'_t(t;1,y)]^2 = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)(2^{k+3}-1)(\ln y)^{k+4}}{(k+4)!} t^k.$$

By standard arguments for series, we can get two combinatorial identities:

$$\sum_{i=0}^{k}(i+1)\binom{k+3}{i+2}=(k+1)(2^{k+2}-1), \tag{2.11}$$

$$\sum_{i=0}^{k}(i+1)(k-i+1)\binom{k+4}{i+2}=2(k+3)(1+k\cdot 2^{k+1}). \tag{2.12}$$

By further computation, the following expansions are obtained

(2.13)

$$g^2(t;1,y)g_t'''(t;1,y)\equiv\sum_{k=0}^{\infty}\alpha_k(\ln y)^{k+6}t^k$$
$$=\sum_{k=0}^{\infty}\frac{2}{(k+6)!}\sum_{i=0}^{k}(i+1)(i+2)(i+3)(2^{k-i+1}-1)\binom{k+6}{i+4}(\ln y)^{k+6}t^k,$$

(2.14)

$$g(t;1,y)g_t'(t;1,y)g_t''(t;1,y)\equiv\sum_{k=0}^{\infty}\beta_k(\ln y)^{k+6}t^k$$
$$=\sum_{k=0}^{\infty}\frac{1}{(k+6)!}\sum_{i=0}^{k}(i+1)(i+2)(k-i+1)(2^{k-i+2}-1)\binom{k+6}{i+3}\big(\ln y\big)^{k+6}t^k,$$

(2.15)

$$g_t'(t;1,y)\big([g_t'(t;1,y)]^2+g(t;1,y)g_t''(t;1,y)\big)\equiv\sum_{k=0}^{\infty}\gamma_k(\ln y)^{k+6}t^k$$
$$=\sum_{k=0}^{\infty}\frac{1}{(k+6)!}\sum_{i=0}^{k}(i+1)(i+2)(k-i+1)(2^{i+3}-1)\binom{k+6}{i+4}\big(\ln y\big)^{k+6}t^k.$$

Proposition 1. *For an arbitrary nonnegative integer k, we have*

$$2\sum_{i=0}^{k}(i+1)(i+2)\big\{(i+3)2^{k-i+1}+(k-i+1)2^{i+3}-(k+4)\big\}\binom{k+6}{i+4}$$
$$\leq 5\sum_{i=0}^{k}(i+1)(i+2)(k-i+1)(2^{k-i+2}-1)\binom{k+6}{i+3}. \tag{2.16}$$

Proof. The inequality (2.16) is equivalent to

$$\sum_{j=0}^{k+6}(j-3)(j-2)\big[(k-j+5)2^j+(j-1)2^{k-j+6}-2(k+4)\big]\binom{k+6}{j}$$
$$+(k^2+12k+38)(2^{k+6}+k+3)$$
$$\leq 5\sum_{j=0}^{k+6}(j-2)(j-1)(k-j+4)(2^{k-j+5}-1)\binom{k+6}{j}. \tag{2.17}$$

Using expansions-into-series of $(1+2x)^n$ and its derivatives, we get

$$\begin{aligned}
&\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n, \qquad \sum_{i=0}^{n} i2^i \binom{n}{i} = 2n3^{n-1},\\
&\sum_{i=0}^{n} i(i-1)2^i \binom{n}{i} = 4n(n-1)3^{n-2},\\
&\sum_{i=0}^{n} i(i-1)(i-2)2^i \binom{n}{i} = 8n(n-1)(n-2)3^{n-3}.
\end{aligned} \tag{2.18}$$

Similarly, using expansions-into-series of $(1+x/2)^n$ and its derivatives, we have

$$\begin{aligned}
&\sum_{j=0}^{n} \frac{1}{2^j} \binom{n}{j} = \Big(\frac{3}{2}\Big)^n, \qquad \sum_{j=0}^{n} \frac{j}{2^j} \binom{n}{j} = \frac{n}{2}\Big(\frac{3}{2}\Big)^{n-1},\\
&\sum_{j=0}^{n} \frac{j(j-1)}{2^j} \binom{n}{j} = \frac{n(n-1)}{4}\Big(\frac{3}{2}\Big)^{n-2},\\
&\sum_{j=0}^{n} \frac{j(j-1)(j-2)}{2^j} \binom{n}{j} = \frac{n(n-1)(n-2)}{8}\Big(\frac{3}{2}\Big)^{n-3}.
\end{aligned} \tag{2.19}$$

The following formulae are also well-known

$$\begin{aligned}
&\sum_{j=0}^{n} \binom{n}{j} = 2^n, \qquad \sum_{j=0}^{n} j\binom{n}{j} = n2^{n-1},\\
&\sum_{j=0}^{n} j(j-1)\binom{n}{j} = n(n-1)2^{n-2},\\
&\sum_{j=0}^{n} j(j-1)(j-2)\binom{n}{j} = n(n-1)(n-2)2^{n-3}.
\end{aligned} \tag{2.20}$$

Substitution of (2.18), (2.19) and (2.20) into (2.17) and simplification give us

$$(k^3+15k^2+74k+114) + (k^3+15k^2+74k+168)2^{k+3} - 2\times 3^{k+6} \le 0. \tag{2.21}$$

From the Taylor's expansion

$$a^x = \sum_{i=0}^{\infty} \frac{(\ln a)^i}{i!} x^i, \tag{2.22}$$

the inequality (2.21) reduces to

$$\begin{aligned}
&1458 + (666 + 1344\ln 2)k + 8\big[84(\ln 2)^2 + 74\ln 2 + 15\big]k^2 + k^3\\
&\quad + 8\sum_{i=3}^{\infty}\left[\frac{168(\ln 2)^3}{i(i-1)(i-2)} + \frac{74(\ln 2)^2}{(i-1)(i-2)} + \frac{15\ln 2}{i-2} + 1\right]\frac{(\ln 2)^{i-3}}{(i-3)!}\cdot k^i\\
&\qquad \le 1458\sum_{i=0}^{\infty}\frac{(\ln 3)^i}{i!}\cdot k^i.
\end{aligned} \tag{2.23}$$

To prove inequality (2.23), by equating the coefficients of k^i in (2.23), it is sufficient to verify that

(2.24)
$$4\big[168(\ln 2)^3 + 74(\ln 2)^2 i + 15i(i-1)\ln 2 + i(i-1)(i-2)\big] \le 729(\ln 2)^3(\log_2 3)^i$$

holds for $i \ge 4$. Since

(2.25)
$$4\big[x^3 + (15\ln 2 - 3)x^2 + \big(2 - 15\ln 2 + 74(\ln 2)^2\big)x + 168(\ln 2)^3\big] \le 729(\ln 2)^3(\log_2 3)^x$$

is valid for $x \ge 0$, inequality (2.24) follows. This completes the proof of Proposition 1. ∎

Corollary 1. *For any nonnegative number $x \ge 0$, we have*

$$\frac{1+3^{x+3}}{1+2^{x+3}} \ge \frac{x^3+15x^2+74x+168}{54}. \tag{2.26}$$

Proposition 2. *If $y > x = 1$, then, for $t \ge 0$,*

$$g^2(t;1,y)g_t'''(t;1,y) - 3g(t;1,y)g_t'(t;1,y)g_t''(t;1,y) + 2[g_t'(t;1,y)]^3 \le 0. \tag{2.27}$$

Proof. It is clear that

$$\begin{aligned} &g^2(t)g'''(t) - 3g(t)g'(t)g''(t) + 2\big[g'(t)\big]^3 \\ =&g^2(t)g'''(t) - 5\big(g(t)g'(t)\big)g''(t) + 2g'(t)\big[g(t)g''(t) + \big(g'(t)\big)^2\big] \\ =&\sum_{k=0}^{\infty}(\alpha_k - 5\beta_k + 2\gamma_k)(\ln y)^{k+6}t^k. \end{aligned} \tag{2.28}$$

Furthermore, Proposition 1 implies

$$\alpha_k - 5\beta_k + 2\gamma_k \le 0, \quad k \ge 0. \tag{2.29}$$

The proof of Proposition 2 is completed. ∎

Theorem 1. *The extended mean values $E(r,s;x,y)$ are logarithmically concave on $(0,+\infty)$ with respect to either r or s, respectively; and logarithmically convex on $(-\infty,0)$ with respect to either r or s, respectively.*

Proof. Combination of Proposition 2 with equality (2.6) figures out that, for $y > x = 1$, $g_t'(t;1,y)/g(t;1,y)$ is concave on $(0,+\infty)$ with t. Therefore, from the Lemma, it follows that the extended mean values $E(r,s;1,y)$ are logarithmically concave on $(0,+\infty)$ with respect to either r or s for $y > x = 1$.

By standard arguments, we obtain

$$\begin{aligned} E(r,s;x,y) &= xE(r,s;1,\frac{y}{x}), \\ E(-r,-s;x,y) &= \frac{xy}{E(r,s;x,y)}. \end{aligned} \tag{2.30}$$

Hence, $E(r,s;x,y)$ are logarithmically concave on $(0,+\infty)$ with either r or s, respectively; and logarithmically convex on $(-\infty,0)$ in either r or s, respectively. The proof of Theorem 1 is completed. ∎

Corollary 2. *The logarithmic means $L(r;x,y) = \big(L(x^r,y^r)\big)^{1/r}$ and extended logarithmic means $S_r(x,y) = E(r,1;x,y)$ are logarithmically concave (or logarithmically convex, respectively) on $(0,+\infty)$ (or on $(-\infty,0)$, respectively) with respect to r.*

Corollary 3. *For any $y > x > 0$, if $t \geq 0$, then*

$$g^2(t;x,y)g_t'''(t;x,y) - 3g(t;x,y)g_t'(t;x,y)g_t''(t;x,y) + 2[g_t'(t;x,y)]^3 \leq 0. \tag{2.31}$$

If $t \leq 0$, the inequality (2.31) *reverses.*

As concrete applications of the logarithmic convexities of $E(r,s;x,y)$, an inequality of mean values is given as follows.

Theorem 2. *Let $A(x,y)$, $L(x,y)$ and $I(x,y)$ denote the arithmetic mean, the logarithmic mean, and the identric mean of two variables x and y. Then, for $x \neq y$, we have*

$$I(x,y) > \frac{L(x,y) + A(x,y)}{2}. \tag{2.32}$$

Proof. P. Montel [4, p. 19] verified that a positive function $f(x)$ is logarithmically convex if and only if $x \longmapsto \mathrm{e}^{ax} f(x)$ is a convex function for all real values of a. Thus, applied this conclusion to the function $\mathrm{e}^{ar}\, E(r,s;x,y)$, for any given $a \in \mathbb{R}$ and $x, y > 0$, $x \neq y$, we get

$$I(x,y) \geq \frac{e^{-a}L(x,y) + e^{a}A(x,y)}{2} \geq \sqrt{A(x,y)L(x,y)}. \tag{2.33}$$

The proof is complete. ∎

3. Miscellanea

A function $f(t)$ is said to be absolutely monotonic on (a,b) if it has derivatives of all orders and $f^{(k)}(t) \geq 0$, $t \in (a,b)$, $k \in \mathbb{N}$.

A function $f(t)$ is said to be completely monotonic on (a,b) if it has derivatives of all orders and $(-1)^k f^{(k)}(t) \geq 0$, $t \in (a,b)$, $k \in \mathbb{N}$.

The famous Bernstein-Widder theorem [18] states that, a function $f(x)$, $x \in (0,+\infty)$ is absolutely monotone if and only if there exists a bounded and nondecreasing function $\sigma(t)$ such that the integral

$$f(x) = \int_0^{+\infty} e^{xt}\, d\sigma(t) \tag{3.1}$$

converges for $0 \leq x < +\infty$; a function $f(x)$, $x \in (0,+\infty)$ is completely monotone if and only if there exists a bounded and nondecreasing function $\eta(t)$ such that the integral

$$f(x) = \int_0^{+\infty} e^{-xt}\, d\eta(t) \tag{3.2}$$

converges for $0 \leq x < +\infty$.

Proposition 3. *Suppose $F(t) = \int_a^b p(u) f^t(u)\, du$, $t \in \mathbb{R}$, $p(u) \not\equiv 0$ is a nonnegative and continuous function, and $f(u)$ a positive and continuous function on a given interval $[a,b]$. Then*

$$F^{(n)}(t) = \int_a^b p(u) f^t(u) \big[\ln f(u)\big]^n\, du. \tag{3.3}$$

If $f(u) \geq 1$, $F(t)$ is absolutely monotone on $(-\infty,+\infty)$; if $0 < f(u) < 1$, then $F(t)$ is completely monotone on $(-\infty,+\infty)$. Moreover, $F(t)$ is absolutely convex on $(-\infty,+\infty)$.

Proof. This is obvious. ∎

Corollary 4 ([13, 14]). *The function $g(t;x,y)$ is absolutely and regularly monotonic on $(-\infty,+\infty)$ for $y>x>1$, or on $(0,+\infty)$ for $y>x^{-1}>1$, completely and regularly monotonic on $(-\infty,+\infty)$ for $0<x<y<1$, or on $(-\infty,0)$ for $1<y<x^{-1}$. Furthermore, $g(x)$ is absolutely convex on $(-\infty,+\infty)$.*

The generalized weighted mean values $M_{p,f}(r,s;x,y)$ with two parameters r and s are defined in [8, 9] by

$$M_{p,f}(r,s;x,y)=\left(\frac{\int_x^y p(u)f^s(u)du}{\int_x^y p(u)f^r(u)du}\right)^{1/(s-r)}, \qquad (r-s)(x-y)\neq 0; \tag{3.4}$$

$$M_{p,f}(r,r;x,y)=\exp\left(\frac{\int_x^y p(u)f^r(u)\ln f(u)du}{\int_x^y p(u)f^r(u)du}\right), \qquad x-y\neq 0; \tag{3.5}$$

$$M_{p,f}(r,s;x,x)=f(x),$$

where $x,y,r,s\in\mathbb{R}$, $p(u)\not\equiv 0$ is a nonnegative and integrable function and $f(u)$ a positive and integrable function on the interval between x and y.

It is clear that $E(r,s;x,y)$ is a special case of $M_{p,f}(r-1,s-1;x,y)$ applied to $p(u)\equiv 1$, $f(u)=u$, and $M^{[r]}(f;p;x,y)=M_{p,f}(r,0;x,y)$.

The basic properties and monotonicities of $M_{p,f}(r,s;x,y)$ were studied in [8, 16, 19].

At last, we propose the following two conjectures

Open Problem 1. *If $f(x)$ is an absolutely or completely monotonic function on the interval $(-\infty,+\infty)$, then the following inequality holds for $0\le x<+\infty$ or reverses for $-\infty<x\le 0$:*

$$f^2(x)f'''(x)-3f(x)f'(x)f''(x)+2\left[f'(x)\right]^3\le 0. \tag{3.6}$$

Open Problem 2. *Suppose $p(u)$ is nonnegative and continuous, and $f(u)$ positive and continuous on a given interval $[a,b]$. If $f(u)\ge 1$ or $0<f(u)<1$, then the generalized weighted mean values $M_{p,f}(r,s;x,y)$ are logarithmically concave on $(0,+\infty)$ with respect to either r or s, respectively; or logarithmically convex on $(-\infty,0)$ with respect to either r or s, respectively.*

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