SOME ERROR ESTIMATES IN THE TRAPEZOIDAL QUADRATURE RULE

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Abstract. We derive sharp error bounds in the Trapezoidal Rule for functions whose first derivative satisfies a strong continuity condition or whose second derivative is an L_p function.

1. Introduction

The classical error estimate in the Trapezoidal Rule asserts that if a function f has a bounded second derivative on the interval $[a,b]$, then the Trapezoidal Rule converges to the true value of the definite integral $\int_a^b f(x)\,dx$ with the rapidity of at least $\frac{1}{n^2}$. If the second derivative of f is *not* bounded, then the Trapezoidal Rule may or may not converge, and if it converges, it may do so very slowly.

In some practical problems the integrand does not have a bounded second derivative, and therefore the classical error estimate of the Trapezoidal Rule is not available. In this paper we derive some error bounds in the Trapezoidal Quadrature Rule for the following classes of functions:

(1) Functions whose first derivative belongs to the Hölder Space $C^{\alpha}[a,b]$, where for $\alpha \in (0,1]$,

$$C^{\alpha}[a,b] := \{f : [a,b] \to \mathbb{R} : |f(x) - f(y)| \le K|x-y|^{\alpha},\ x,y \in [a,b]\}$$

for some finite $K > 0$,

(2) Functions whose second derivative is an L_p-function.

This paper is organized as follows: In this section we prove the key lemma (Lemma 1). This lemma is then applied to the classes of functions mentioned above.

In Section 2 we apply the results of Section 1 in deriving some inequalities of Trapezoid type.

In Section 3 we apply the results obtained in Section 2 to derive error bounds for the Trapezoidal Quadrature Rule for the above classes of functions.

Unless otherwise indicated, we shall assume throughout that $[a,b]$ is a finite interval.

The following lema will be useful in the sequel.

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Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then we have the identity*

$$\begin{aligned} & \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx \\ = & \frac{1}{2(b-a)^2}\int_a^b\int_a^b [f'(x)-f'(y)](x-y)\,dxdy. \end{aligned}$$

Proof. We have successively

$$\begin{aligned} & \int_a^b\int_a^b [f'(x)-f'(y)](x-y)\,dxdy \\ = & \int_a^b\int_a^b [xf'(x)+yf'(y)-xf'(y)-yf'(x)]\,dxdy \\ = & 2\int_a^b\int_a^b [xf'(x)-xf'(y)]\,dxdy \\ = & 2\int_a^b\int_a^b xf'(x)\,dxdy-2\int_a^b\int_a^b xf'(y)\,dxdy \\ = & 2(b-a)\left[bf(b)-af(a)-\int_a^b f(x)\,dx\right]-\left(b^2-a^2\right)[f(b)-f(a)] \\ = & (b-a)^2[f(a)+f(b)]-2(b-a)\int_a^b f(x)\,dx. \end{aligned}$$

Dividing both sides by $2(b-a)^2$ yields the required result. ∎

2. Some Integral Inequalities

Let us start to the following result for convex mappings.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\geq\max\{|A|,|B|,|C|\}, \tag{2.1}$$

where

$$\begin{aligned} A & : =\frac{1}{b-a}\int_a^b |xf'(x)|\,dx-\frac{1}{(b-a)^2}\int_a^b |f'(x)|\,dx\cdot\int_a^b |x|\,dx, \\ B & : =\frac{1}{b-a}\int_a^b x\,|f'(x)|\,dx-\frac{a+b}{2}\cdot\frac{1}{b-a}\int_a^b |f'(x)|\,dx, \text{ and} \\ C & : =\frac{1}{b-a}\int_a^b |x|\,f'(x)\,dx-\frac{f(b)-f(a)}{(b-a)^2}\int_a^b |x|\,dx. \end{aligned}$$

Moreover, the inequality (2.1) *is sharp in the sense that it cannot be replaced by a smaller constant.*

Proof. Since f is a convex function on $[a,b]$, it follows that

$$[f'(x)-f'(y)](x-y)\geq 0$$

for almost every $x, y \in (a,b)$. Hence, for a.e. $x, y \in (a,b)$,

$$\begin{aligned}
& [f'(x) - f'(y)](x-y) \\
= \; & |[f'(x) - f'(y)](x-y)| \\
\geq \; & \max\{|[|f'(x)| - |f'(y)|][|x| - |y|]|, |[|f'(x)| - |f'(y)|][x-y]|, \\
& |[f'(x) - f'(y)][|x| - |y|]|\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \int_a^b \int_a^b [f'(x) - f'(y)][x-y]\,dxdy \\
\geq \; & \max\left\{\left|\int_a^b \int_a^b [|f'(x)| - |f'(y)|][|x| - |y|]\,dxdy\right|,\right. \\
& \left|\int_a^b \int_a^b [|f'(x)| - |f'(y)|][x-y]\,dxdy\right|, \\
& \left.\left|\int_a^b \int_a^b [f'(x) - f'(y)][|x| - |y|]\,dxdy\right|\right\}.
\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}
A &= \frac{2}{(b-a)^2}\int_a^b \int_a^b [|f'(x)| - |f'(y)|][|x| - |y|]\,dxdy, \\
B &= \frac{2}{(b-a)^2}\int_a^b \int_a^b [|f'(x)| - |f'(y)|][x-y]\,dxdy, \text{ and} \\
C &= \frac{2}{(b-a)^2}\int_a^b \int_a^b [f'(x) - f'(y)][|x| - |y|]\,dxdy.
\end{aligned}$$

To prove the sharpness of the inequality (2.1), let $f(x) = \frac{1}{x}$. Then f is differentiable and convex on (a,b), where $a > 0$. Now,

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{a+b}{2ab} - \left(\frac{\ln b - \ln a}{b-a}\right),$$

and

$$\begin{aligned}
|B| &= \left|\frac{1}{b-a}\int_a^b \frac{1}{x}dx - \frac{a+b}{2}\cdot\frac{1}{b-a}\int_a^b \frac{1}{x^2}dx\right| \\
&= \left|\frac{\ln b - \ln a}{b-a} - \frac{a+b}{2ab}\right| = \frac{a+b}{2ab} - \left(\frac{\ln b - \ln a}{b-a}\right).
\end{aligned}$$

We therefore get equality in the inequality (2.1). ∎

The following inequality also holds.

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f' \in C^{\alpha}[a,b]$, then we have the inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{K}{(\alpha+2)(\alpha+3)}(b-a)^{\alpha+1},$$

for some finite constant $K > 0$.

Proof. Since $f' \in C^{\alpha}[a,b]$, there is a finite constant $K > 0$ such that

$$|f'(x) - f'(y)| \le K|x-y|^{\alpha+1} \text{ for all } x, y \in [a,b].$$

Using Lemma 1, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\le\ & \frac{1}{2(b-a)^2}\int_a^b\int_a^b |f'(x)-f'(y)|\,|x-y|\,dxdy \\
\le\ & \frac{K}{2(b-a)^2}\int_a^b\int_a^b |x-y|^{\alpha+1}\,dxdy \\
=\ & \frac{K}{(\alpha+2)(\alpha+3)}(b-a)^{\alpha+1},
\end{aligned}$$

where the last equality follows from the fact that

$$\begin{aligned}
\int_a^b\int_a^b |x-y|^{\alpha+1}\,dxdy &= \int_a^b\left(\int_y^b (x-y)^{\alpha+1}\,dx + \int_a^y (y-x)^{\alpha+1}\,dx\right)dy \\
&= \frac{1}{\alpha+2}\int_a^b\left[(b-y)^{\alpha+2} + (y-a)^{\alpha+2}\right]dy \\
&= \frac{2(b-a)^{\alpha+3}}{(\alpha+2)(\alpha+3)}.
\end{aligned}$$

and the theorem is proved. ∎

The following corollary is natural.

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If f' is Lipschitz continuous on $[a,b]$, i.e.,*

$$|f'(x) - f'(y)| \le K|x-y| \text{ for all } x, y \in [a,b],$$

where $K > 0$, then we have the inequality

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{K}{12}(b-a)^2.$$

Moreover, the constant $\frac{1}{12}$ is sharp.

Proof. Take $\alpha = 1$ in Theorem 2.
To prove the sharpness of the constant $\frac{1}{12}$, let $f(x) = \frac{x^2}{2}$. Then

$$|f'(x) - f'(y)| = |x-y| \le K|x-y|$$

for all $x \in [a,b]$, where $K = 1$. Consequently,

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{12}(b-a)^2.$$

and the proof is completed. ∎

The following theorem also holds.

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be such that f' is absolutely continuous on $[a,b]$. If $f''\in L_p(a,b)$, where $1\le p<\infty$, then we have the estimation*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \le\;& \frac{1}{2(b-a)^2}\int_a^b\int_a^b K_p(x,y)\,|x-y|^{2-\frac{1}{p}}\,dxdy\\ \le\;& \frac{p^2(b-a)^{2-\frac{1}{p}}}{(3p-1)(4p-1)}\,\|f''\|_p\,,\end{aligned}$$

where

$$K_p(x,y)=\left|\int_x^y |f''(t)|^p\,dt\right|^{\frac{1}{p}},\;\; x,y\in[a,b].$$

The first inequality is sharp.

Proof. By Lemma 1,

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\\ =\;& \frac{1}{2(b-a)^2}\int_a^b\int_a^b [f'(x)-f'(y)](x-y)\,dxdy\\ =\;& \frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(\int_x^y f''(\eta)\,d\eta\right)(x-y)\,dxdy.\end{aligned}$$

Also, using Hölder's Inequality for $p>1, 1/p+1/q=1$, we have that, for all $x,y\in[a,b]$,

$$\left|\int_x^y |f''(\eta)|\,d\eta\right|\le\left|\int_x^y |f''(\eta)|^p\,d\eta\right|^{\frac{1}{p}}\left|\int_x^y d\eta\right|^{\frac{1}{q}}=|x-y|^{\frac{1}{q}}K_p(x,y).$$

Now,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\;& \frac{1}{2(b-a)^2}\int_a^b\int_a^b\left|\int_x^y f''(\eta)\,d\eta\right||y-x|\,dxdy \\
\leq\;& \frac{1}{2(b-a)^2}\int_a^b\int_a^b|x-y|^{1-\frac{1}{q}}\left|\int_x^y|f''(\eta)|^p\,d\eta\right|^{\frac{1}{p}}dxdy \\
=\;& \frac{1}{2(b-a)^2}\int_a^b\int_a^b|x-y|^{2-\frac{1}{p}}\left|\int_x^y|f''(t)|^p\,dt\right|^{\frac{1}{p}}dxdy \\
\leq\;& \frac{1}{2(b-a)^2}\int_a^b\int_a^b|x-y|^{2-\frac{1}{p}}\left(\int_a^b|f''(t)|^p\,dt\right)^{\frac{1}{p}}dxdy \\
=\;& \frac{\|f''\|_p}{2(b-a)^2}\cdot\frac{2p^2(b-a)^{4-\frac{1}{p}}}{(3p-1)(4p-1)} \\
=\;& \frac{p^2(b-a)^{2-\frac{1}{p}}}{(3p-1)(4p-1)}\|f''\|_p\,.
\end{aligned}$$

To prove the sharpness of the first inequality, let $f(x)=\frac{x^2}{2}$, for $x\in[a,b]$. Then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|=\frac{(b-a)^2}{12},$$

and

$$\begin{aligned}
&\frac{1}{2(b-a)^2}\int_a^b\int_a^b K_p(x,y)\,|x-y|\,dxdy \\
=\;& \frac{1}{2(b-a)^2}\int_a^b\int_a^b|x-y|^{\frac{1}{p}}\,|x-y|^{2-\frac{1}{p}}\,dxdy \\
=\;& \frac{1}{2(b-a)^2}\int_a^b\int_a^b|x-y|^2\,dxdy \\
=\;& \frac{(b-a)^2}{12}.
\end{aligned}$$

The case $p=1$ goes likewise and we omit the details. ∎

3. Error Bounds in the Trapezoidal Quadrature Rule

In this section we derive error bounds in the Trapezoidal Quadrature Rule for the classes of functions mentioned in Section 1.

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function such that $f'\in C^\alpha[a,b]$. If $P:a=x_0<x_1<x_2<...<x_n=b$ is a partition of the interval $[a,b]$, $h_i=x_i-x_{i-1}$ for $i=1,2,...,n$, and*

$$T_n(f,P):=\sum_{i=1}^n\frac{f(x_{i-1})+f(x_i)}{2}\cdot h_i$$

then

$$|R_n(f,P)| = \left|\int_a^b f(x)\,dx - T_n(f,P)\right| \le \frac{K}{(\alpha+2)(\alpha+3)}\sum_{i=1}^{n} h_i^{\alpha+2}$$

for some finite constant $K>0$.

Proof. We just have to apply Theorem 2 to f on each subinterval $[x_{i-1},x_i]$, for $i=1,2,3,...,n$. This gives

$$\begin{aligned}
|R_n(f,P)| &= \left|\int_a^b f(x)\,dx - T_n(f,P)\right| \\
&= \left|\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(x)\,dx - \sum_{i=1}^{n}\frac{f(x_{i-1})+f(x_i)}{2}h_i\right| \\
&= \left|\sum_{i=1}^{n} h_i\left(\frac{f(x_{i-1})+f(x_i)}{2} - \frac{1}{h_i}\int_{x_{i-1}}^{x_i} f(x)\,dx\right)\right| \\
&\le \sum_{i=1}^{n} h_i\left|\frac{f(x_{i-1})+f(x_i)}{2} - \frac{1}{h_i}\int_{x_{i-1}}^{x_i} f(x)\,dx\right| \le \\
&\le \sum_{i=1}^{n} h_i\frac{K}{(\alpha+2)(\alpha+3)}(x_i-x_{i-1})^{\alpha+1} \\
&= \frac{K}{(\alpha+2)(\alpha+3)}\sum_{i=1}^{n} h_i^{\alpha+2}.
\end{aligned}$$

and the theorem is completely proved. ∎

The following three corollaries are natural consequences.

Corollary 2. *Assume that the hypotheses of Theorem 4 hold and that* P *is a regular partition of* $[a,b]$, *i.e.,* $x_i = a + i\left(\frac{b-a}{n}\right)$ *for* $i=0,1,2,...,n$. *Then*

$$|R_n(f,P)| \le \frac{K(b-a)^{\alpha+2}}{(\alpha+2)(\alpha+3)n^{\alpha+1}}$$

for some finite constant $K>0$.
Moreover, if $\varepsilon>0$ *is given, then*

$$n_\varepsilon := \left[\left(\frac{K(b-a)^{\alpha+2}}{(\alpha+2)(\alpha+3)\varepsilon}\right)^{\frac{1}{\alpha+1}}\right]+1$$

where, for $r\in\mathbb{R}$, $[r]$ *is the integer part of* r, *is the smallest natural number for which we have*

$$|R_n(f)| \le \varepsilon \text{ for } n \ge n_\varepsilon.$$

The second part of Corollary 2 says that if we use a regular (or uniform) partition of $[a,b]$ and we want the magnitude of the error to be less than some preassigned tolerance ε, then we need to use at least

$$n_\varepsilon := \left[\left(\frac{K(b-a)^{\alpha+2}}{(\alpha+2)(\alpha+3)\varepsilon}\right)^{\frac{1}{\alpha+1}}\right]+1$$

partition points.

Corollary 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function such that f' is Lipschitz continuous on $[a,b]$. If $P: a=x_0<x_1<x_2<...<x_n=b$ is a partition of the interval $[a,b]$, $h_i=x_i-x_{i-1}$ for $i=1,2,...,n$, and*

$$T_n(f,P):=\sum_{i=1}^{n}\frac{f(x_{i-1})+f(x_i)}{2}\cdot h_i$$

Then

$$|R_n(f,P)|\leq\frac{K}{12}\sum_{i=1}^{n}h_i^3.$$

Moreover, the constant $\frac{1}{12}$ is sharp.

Corollary 4. *Assume that the hypotheses of Corollary 3 hold and that P is a regular partition of $[a,b]$, i.e., $x_i=a+i\left(\frac{b-a}{n}\right)$ for $i=0,1,2,...,n$. Then*

$$|R_n(f,P)|=\left|\int_a^b f(x)\,dx-T_n(f,P)\right|\leq\frac{L(b-a)^3}{12n^2}.$$

Moreover, if $\varepsilon>0$ is given, then

$$n_\varepsilon:=\left[\sqrt{\frac{L(b-a)^3}{12\varepsilon}}\right]+1$$

is the smallest natural number for which we have

$$|R_n(f)|\leq\varepsilon \text{ for } n\geq n_\varepsilon.$$

In the case of functions which are twice differentiable and the second derivative is integrable, we have the following error estimate.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b) such that $f''\in L_1(a,b)$ and $P: a=x_0<x_1<x_2<...<x_n=b$ is a partition of the interval $[a,b]$. Then we have the following estimation*

$$\begin{aligned}(3.1)\qquad & |R_n(f,P)| \\ \leq\ & \frac{1}{2}\sum_{i=1}^{n}\frac{1}{h_i}\int_{x_{i-1}}^{x_i}\int_{x_{i-1}}^{x_i}K_1(x,y)\,|x-y|\,dxdy \\ \leq\ & \frac{\|P\|^2}{6}\|f''\|_1,\end{aligned}$$

where $K_1(x,y)=\left|\int_x^y|f''(t)|\,dt\right|$ with $x,y\in[a,b]$, and $\|P\|=\max_{1\leq i\leq n}\{x_i-x_{i-1}\}$. The first inequality in (3.1) is sharp.

Proof. Applying Theorem 3 on each subinterval $[x_{i-1}, x_i]$ for $i = 1, 2, ..., n,$ we obtain

$$
\begin{aligned}
& |R_n(f,P)| \\
= & \left| \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx - \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2} h_i \right| \\
= & \left| \sum_{i=1}^{n} h_i \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} f(x)\,dx - \frac{f(x_{i-1}) + f(x_i)}{2} \right) \right| \\
\leq & \sum_{i=1}^{n} h_i \left| \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} f(x)\,dx - \frac{f(x_{i-1}) + f(x_i)}{2} \right) \right| \\
\leq & \sum_{i=1}^{n} h_i \frac{1}{2h_i^2} \int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} K_1(x,y) |x-y|\,dxdy \\
\leq & \sum_{i=1}^{n} \frac{h_i^2}{6} \int_{x_{i-1}}^{x_i} |f'(t)|\,dt \\
\leq & \frac{\|P\|^2}{6} \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f''(t)|\,dt \\
\leq & \frac{\|P\|^2}{6} \|f''\|_1,
\end{aligned}
$$

which completes the proof. ∎

Assume that the hypotheses of Theorem 5 hold and that P is a regular partition of $[a,b]$. Then

$$|R_n(f,P)| \leq \frac{(b-a)^2}{6n^2} \|f''\|_1.$$

If $\varepsilon > 0$ is given, then

$$n_\varepsilon := \left[(b-a) \sqrt{\frac{\|f''\|_1}{6\varepsilon}} \right] + 1$$

is the smallest natural number for which we have

$$|R_n(f)| \leq \varepsilon \text{ for } n \geq n_\varepsilon.$$

Finally, we have

Theorem 6. *Let $f : [a,b] \to R$ be so that f' is an absolutely continuous function on (a,b) and such that $f'' \in L_p(a,b)$, where $1 < p < \infty$, and $P : a = x_0 < x_1 < x_2 < ... < x_n = b$ is a partition of the interval $[a,b]$. Then we have the following estimation*

$$
\begin{aligned}
& |R_n(f,P)| \\
\leq & \frac{1}{2} \sum_{i=1}^{n} \frac{1}{h_i} \int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} K_p(x,y) |x-y|^{2-\frac{1}{p}}\,dxdy \\
\leq & \frac{p^2}{(3p-1)(4p-1)} \left(\sum_{i=1}^{n} h_i^{\frac{2p-1}{p-1}} \right)^{1-\frac{1}{p}} \|f''\|_p,
\end{aligned}
$$

where

$$K_p(x,y) = \left| \int_x^y |f''(t)|^p \, dt \right|^{\frac{1}{p}}, \; x, y \in [a,b].$$

The first inequality is sharp.

A naturally corollary of this theorem is

Corollary 5. *Assume that the hypothesis of Theorem 6 hold and that P is a regular partition of $[a,b]$. Then*

$$|R_n(f,P)| \leq \frac{p^2 (b-a)^{2-\frac{1}{p}}}{n^2 (3p-1)(4p-1)} \|f''\|_p .$$

If $\varepsilon > 0$ is given, then

$$n_\varepsilon := \left[p \sqrt{\frac{(b-a)^{2-\frac{1}{p}} \|f''\|_p}{(3p-1)(4p-1)\varepsilon}} \right] + 1$$

is the smallest natural number for which we have

$$|R_n(f)| \leq \varepsilon \text{ for } n \geq n_\varepsilon.$$

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