

JENSEN-TYPE INEQUALITIES FOR INVEX FUNCTIONS

B.D. CRAVEN AND S.S. DRAGOMIR

ABSTRACT. Jensen's inequality for a real convex function f on a convex domain is generalised in several ways to vector functions with a cone inequality, and to invex functions generalizing convex functions.

1. INTRODUCTION

Jensen's inequality for a real convex function f on a convex domain C states that, whenever $x_1, x_2, \dots \in C$ and $\alpha_1, \alpha_2, \dots \geq 0$ with $\sum \alpha_i = 1$,

$$f\left(\sum \alpha_i x_i\right) \leq \sum \alpha_i f(x_i).$$

This paper presents several generalisations of this inequality, to vector functions F with a cone inequality, and to invex functions generalizing convex functions. This introduces additional gradient terms into various inequalities.

2. BASIC CONCEPTS AND DEFINITIONS

Definition 1. Let X and Z be normed spaces, $Q \subset Z$ a closed convex cone, and $E \subset X$ a convex open set. A function $F : E \rightarrow Z$ is Q -convex on E if

$$(\forall x, y \in E) (\forall \alpha \in (0, 1)) \quad \alpha F(x) + (1 - \alpha) F(y) - F(\alpha x + (1 - \alpha)y) \in Q.$$

A (Fréchet or linear Gateaux) differentiable function $F : E \rightarrow Z$ is Q -invex (see [1], [2], [3]) if, for some scale function $\omega : E \times E \rightarrow X$,

$$(\forall x, y \in E) \quad F(x) - F(y) - F'(y)\omega(x - y, y) \in Q.$$

Remark 1. $F'(y)$ denotes the derivative. Invex properties have been extensively used with optimization problems. It is well-known that Q -invex implies Q -convex if $(\forall x, y \in E) \omega(x - y, y) = x - y$. If F is real-valued and $Q = \mathbb{R}_+$, then the usual convexity follows. From Q -convex there follows readily

$$(\forall x_i \in E) \left(\forall \alpha_i \geq 0, \sum \alpha_i = 1 \right) \sum \alpha_i F(x_i) - F\left(\sum \alpha_i x_i\right) \in Q.$$

In this paper, invex shall require the function to be differentiable. (Extensions to nondifferentiable functions will be considered elsewhere.)

Denote by \geq_Q the ordering defined by Q thus $b \geq_Q a \Leftrightarrow b - a \in Q$.

If $\omega(\cdot, y)$ is linear, then $(\forall z = x - y) F(y + z) - F(z) \geq_Q F'(y) Cz$, where C is a linear mapping (that may depend on y), hence $(\forall z) F'(y) z \geq_Q F'(y) Cz$;

if Q is pointed (thus if $Q \cap (-Q) = \{0\}$) then F is convex, since

$$(\forall x) F(x) - F(y) \geq_Q F'(y)(x - y) + F'(y)(C - I)(x - y) = F'(y)(x - y).$$

Thus functions where $\omega(\cdot, y)$ is linear for each y are equivalent to convex functions.

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Notation 1. Various results require a suitable notation, in order that the concept is not obscured by algebraic details. Let $x = (x_1, x_2, \dots, x_n)$, where each x_i is a vector in E . Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ where each $\alpha_i \geq 0$ and $\sum \alpha_i = 1$. Denote by $\alpha \cdot x$ the inner product $\sum \alpha_i x_i$, and similarly for other inner products. Denote by S a sequence of indices (i_1, i_2, \dots, i_r) taken from $\{1, 2, \dots, n\}$ with repetitions allowed; similarly denote by S' a sequence of indices $\{i_1, i_2, \dots, i_{r+1}\}$. Denote by x_S the vector of x_i when i runs through S ; and similarly define α_S . Similarly $F'(x_S)$ denotes the sequence of $F'(x_i)$ as i runs through S . Denote by M_{x_S} the mean of the sequence x_S , and by Π_{α_S} the product of the items in α_S . If $\Phi(S)$ denotes a function of S , denote by $A\Phi(S)$ the sum of the items in $\Phi(S)$, when S runs over all sequences of length r , and denote by $A\Phi(S')$ the sum of the items in $\Phi(S')$, where S' runs over the sequences of length $r+1$. Thus $A(\Pi_{\alpha_S})\Phi(S)$ represents the sum of all terms

$$\alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_r} \Phi(i_1, i_2, \dots, i_r)$$

as i_1, i_2, \dots, i_r run through $1, 2, \dots, n$.

Proposition 1. Let $F : E \rightarrow Z$ be Q -invex; let $x_1, x_2, \dots \in E$ (allowing possible repetitions), and let $\alpha_1, \alpha_2, \dots$ be nonnegative with sum 1; let $w = \sum \alpha_i x_i$. Then

$$\sum \alpha_i F(x_i) - F(w) \geq_Q F'(w) \sum \omega(x_i - w, w).$$

Proof. Q -invex requires, for each i , that

$$F(x_i) - F(w) \geq_Q F'(w) \omega(x_i - w, w).$$

Multiplication by α_i and summing gives the result. ■

Proposition 2. Let $F : E \rightarrow Z$ be differentiable Q -invex. Let $x_1, x_2, \dots \in E$ (allowing possible repetitions), and let $\alpha_1, \alpha_2, \dots$ be nonnegative with sum 1. Then

$$\begin{aligned} & - \sum_j \alpha_j F'(x_j) \omega(\alpha \cdot (x - x_j), x_j) \\ & \geq_Q \alpha \cdot F(x_S) - F(\alpha \cdot x) \\ & \geq_Q F'(\alpha \cdot x) \sum \alpha_j \omega(\alpha \cdot (x_j - x), \alpha \cdot x). \end{aligned}$$

Proof. Let $w = \alpha \cdot x$. From invex,

$$F'(x_j) \omega(x - x_j, x_j) \geq_Q F(x_j) - F(w) \geq_Q F'(w) \omega(x_i - w, w).$$

Multiplication by α_j and summing gives the result. ■

Corollary 1. Assume also that E is a linear subspace, and $\omega(\cdot, w)$ is linear. Then $\alpha \cdot F(x_S) - F(\alpha \cdot x) \geq_Q 0$.

Proof. $\sum_j \alpha_j \omega(\alpha \cdot (x - x_j), \alpha \cdot x) = \omega\left(\sum_{j,i} \alpha_j \alpha_i (x_i - x_j), \alpha \cdot x\right) = 0$. ■

Corollary 2. Under hypothesis of Corollary 1,

$$\begin{aligned} & \sum_j \alpha_j F'(x_j) \omega(x_j, x_j) - \sum_{i,j} \alpha_i \alpha_j F'(x_j) \omega(x_j, x_j) \\ & \geq_Q \sum_j \alpha_j F(x_j) - F\left(\sum_j \alpha_j x_j\right) \geq_Q 0. \end{aligned}$$

Proof. Since $\omega(\cdot, w)$ is linear,

$$\begin{aligned} & -\sum_j \alpha_j F'(x_j) \omega(\alpha \cdot (x - x_j), x_j) \\ &= -\sum_j F'(x_j) \omega\left(\sum_i \alpha_j \alpha_i \cdot (x_i - x_j), x_j\right) \\ &= \sum_j \alpha_j F'(x_j) \omega(x_j, x_j) - \sum_{i,j} \alpha_i \alpha_j F'(x_j) \omega(x_j, x_j). \end{aligned}$$

Then the result follows from Corollary 1. ■

Corollary 3. *If F is Q -convex, then*

$$\begin{aligned} & \sum_j \alpha_j F'(x_j) \left(\sum_i \alpha_i x_i\right) - \sum_j \alpha_j F'(x_j) \left(\sum_i \alpha_i x_i\right) \\ & \geq_Q \sum_j \alpha_j F(x_j) - F\left(\sum_j \alpha_j x_j\right) \geq_Q 0. \end{aligned}$$

Proof. By substituting $\omega(\alpha \cdot (x - x_j), x_j) = \sum_j \alpha_j (x_i - x_j)$ into the left inequality of Proposition 2, then rearranging the terms. ■

3. GENERALIZED JENSEN INEQUALITIES FOR INVEX FUNCTIONS

The following generalizations of Jensen's inequalities, involving multiple summations, hold for Q -invex functions.

Proposition 3. *Let $F : E \rightarrow Z$ be differentiable Q -invex; let $r \geq 1$. Then*

$$\begin{aligned} & -A(\Pi\alpha_S) F'(M_{x_S}) \omega((\alpha \cdot (x - M_{x_S}), M_{x_S})) \\ & \geq_Q A(\Pi\alpha_S) F(M_{x_S}) - F(\alpha \cdot x) \\ & \geq_Q F'(\alpha \cdot x) A(\Pi\alpha_S) \omega(\alpha \cdot (M_{x_S} - x), M_{x_S}). \end{aligned}$$

Proof. By the Q -invex property,

$$\begin{aligned} & -F'(M_{x_S}) \omega(\alpha \cdot (x - M_{x_S}), M_{x_S}) \\ & \geq_Q F(M_{x_S}) - F(\alpha \cdot x) \\ & \geq_Q F'(\alpha \cdot x) \omega((M_{x_S} - \alpha \cdot x), \alpha \cdot x). \end{aligned}$$

The result follows by applying $J \equiv A^r \Pi\alpha_S$ on the left, noting that cone inequalities are thus unchanged, and $J\xi = \xi$ for any argument ξ independent of S . ■

Corollary 4. *Assume the hypothesis of Proposition 3. If E is a linear subspace and $\omega(\cdot, M_{x_S})$ is linear, then*

$$A(\Pi\alpha_S) F(M_{x_S}) - F(\alpha \cdot x) \geq_Q 0.$$

Proof. From the linear assumption,

$$A(\Pi\alpha_S) \omega(\alpha \cdot (M_{x_S} - x), M_{x_S}) = \omega(u, M_{x_S}),$$

where, after simplification,

$$u = (A(\Pi\alpha_S)) \alpha \cdot (M_{x_S} - x) = \alpha \cdot (M_{x_S} - x) = 0.$$

■

In the following Corollary, which generalizes Corollary 2, note that $\xi = x_{i_1}$ in the first summation, and S has r terms; in the second summation, M_{x_S} is unchanged, but S' has $r + 1$ terms, and $\eta = x_{i_{r+1}}$.

Corollary 5. *Under the hypothesis of Corollary 4,*

$$\begin{aligned} & -A(\Pi\alpha_S)F'(M_{x_S})\omega(\xi, M_{x_S}) - A(\Pi\alpha_{S'})F'(M_{x_S})\omega(\eta, M_{x_S}) \\ & \geq_Q A(\Pi\alpha_S)F(M_{x_S}) - F(\alpha \cdot x). \end{aligned}$$

Proof. Since $\omega(\cdot, M_{x_S})$ is linear,

$$\begin{aligned} & -A(\Pi\alpha_S)F'(M_{x_S})\omega(\alpha \cdot (\xi - M_{x_S}), M_{x_S}) \\ & = A(\Pi\alpha_S)F'(M_{x_S})\omega(M_{x_S}, M_{x_S}) - A(\Pi\alpha_S)F'(M_{x_S})\alpha \cdot \omega(\xi, M_{x_S}) \\ & = A(\Pi\alpha_S)F'(M_{x_S})\omega(\xi, M_{x_S}) - A(\Pi\alpha_{S'})F'(M_{x_S})\omega(\eta, M_{x_S}). \end{aligned}$$

The conclusion follows by applying Proposition 3. ■

Corollary 6. *If F is convex, then (denoting $\xi = x_{i_1}$)*

$$\begin{aligned} & -A(\Pi\alpha_S)F'(M_{x_S})\xi - A(\Pi\alpha_S)F'(M_{x_S})(M_{x_S}) \\ & \geq_Q A(\Pi\alpha_S)F(M_{x_S}) - F(\alpha \cdot x) \\ & \geq_Q 0. \end{aligned}$$

Proof. Substitute $\omega(\alpha \cdot (x - M_{x_S}), M_{x_S}) = \alpha(x - M_{x_S})$. The details are omitted. ■

4. FURTHER REFINEMENTS

The notation of Corollary 5 is used to state Proposition 4, namely $\xi = x_i$, $\eta = x_{i_{r+1}}$. Note that S has r terms, S' has $r + 1$ terms.

Proposition 4. *Let $F : E \rightarrow Z$ be differentiable Q -invex. Then, with $\varepsilon = (r + 1)^{-1}$,*

$$\begin{aligned} & -A(\Pi\alpha_{S'})F'(M_{x_S})\omega(\varepsilon(\eta - M_{x_S}), M_{x_S}) \\ & \geq_Q A(\Pi\alpha_S)F'(M_{x_S}) - A(\Pi\alpha_{S'})F'(M_{x_{S'}}) \\ & \geq_Q A(\Pi\alpha_{S'})F'(M_{x_{S'}})\omega(\varepsilon(M_{x_S} - \eta), M_{x_{S'}}). \end{aligned}$$

Proof. From Q -invex,

$$\begin{aligned} & F'(M_{x_S})\omega(\varepsilon(\eta - M_{x_S}), M_{x_S}) \\ & \geq_Q F(M_{x_S}) - F(M_{x_{S'}}) \\ & \geq_Q F'(M_{x_{S'}})\omega(\varepsilon(M_{x_S} - \eta), M_{x_{S'}}). \end{aligned}$$

The result follows on applying $(\Pi\alpha_{S'})$ and averaging with A . ■

Corollary 7. *If F is as in Proposition 4, and $\omega(\cdot, M_{x_{S'}})$ is linear, then, for each $r = 1, 2, \dots$,*

$$A(\Pi\alpha_S)F(M_{x_S}) \geq_Q A(\Pi\alpha_{S'})F(M_{x_{S'}}).$$

Proof. This follows from the second inequality of Proposition 4, noting that, when $\omega(\cdot, M_{x_{S'}})$ is linear,

$$\omega(\varepsilon(M_{x_S} - \eta), M_{x_{S'}}) = \left(\frac{\varepsilon}{r}\right) \sum_{j=1}^n \omega(x_{i_j}, M_{x_{S'}}) = \varepsilon\omega(x_{i_{r+1}}, M_{x_{S'}}),$$

and application of $A(\Pi\alpha_{S'})$ gives an equal value for each summand, so the result is 0. ■

Corollary 8. *Under the same hypotheses as Corollary 7,*

$$\begin{aligned} & A(\Pi\alpha_S) F'(M_{x_S}) \xi - A(\Pi\alpha_S) F'(M_{x_S}(M_{x_S})) \\ & \geq_Q A(\Pi\alpha_S) F(M_{x_S}) - A(\Pi\alpha_{S'}) F(M_{x_{S'}}) \\ & \geq_Q 0. \end{aligned}$$

Proof. From the first inequality of Proposition 4, using the assumed linearity to expand $\omega(\cdot, M_{x_S})$ in its first argument. ■

For other recent results connected with Jensen's discrete inequality, see the papers [4]-[9], where further references are given

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MELBOURNE, PARKVILLE, VICTORIA 3052., AUSTRALIA

E-mail address: B.Craven@ms.unimelb.edu.au

SCHOOL OF COMMUNICATIONS AND INFORMATICS, VICTORIA UNIVERSITY OF TECHNOLOGY, PO BOX 14428, MELBOURNE CITY MC 8001, VICTORIA, AUSTRALIA

E-mail address: sever@matilda.vu.edu.au