

AN OSTROWSKI TYPE INEQUALITY IN TWO DIMENSIONS USING THE THREE POINT RULE

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ABSTRACT. An Ostrowski Type inequality in two dimensions for double integrals on a rectangle is developed.

The resulting integral inequalities are valid for the class of functions with bounded first derivatives. They are employed to approximate the double integral by up to 6 one dimensional integrals and nine functions evaluations. Examples using the resulting cubature formulae are presented.

Key words: Cubature, Integral inequality, Ostrowski

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1 Introduction

Cubature formulae are most often evaluated as iterated one-dimensional integrals. The approach is straightforward but has some disadvantages. Two of which are that the error estimates are unnecessarily large, since they too rely on embedding the one-dimensional error results, and it is often difficult to discretize regions that are other than ideal. That is, regions whose boundaries lie on coordinate lines of some orthogonal system.

In this paper we develop a three point cubature rule for two-dimensional rectangular regions. An *a priori* error bound is obtained for functions whose first partial derivatives exist and are bounded. The term “three point” is used to draw an analogy with Newton-Cotes type rules where sampling occurs at the boundary and interior points. The rule presented here approximates a two-dimensional integral via application of function evaluations and one-dimensional integrals at the boundary and interior points. That is, up to nine

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function points and six one-dimensional integrals. A parameterization, similar to that of [2], is employed to distinguish rule type. If the one-dimensional integrals are not known, they themselves can be approximated by a three point rule, as in [2], to produce a cubature rule consisting only of sampling points. As a result the error bound will be larger.

The method presented here is based on Ostrowski's integral inequality, and as such is amenable to the production of error bounds for a variety of norms. This has been done for one dimensional integrals, see for example [2, 3, 4, 5, 6]. The results can be extended to weighted mappings by employing the methods in [9, 7].

2 The Ostrowski Inequality

The classical Ostrowski integral inequality in one dimension stipulates a bound between a function evaluated at an interior point x and the average of the function of over an interval. That is,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty \quad (2.1)$$

for all $x \in [a, b]$, provided that $f' \in L_\infty(a, b)$ and $f : [a, b] \rightarrow \mathbb{R}$ is a differentiable mapping on (a, b) .

Here, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant. We also make the observation that the smallest bound is obtained at $x = \frac{a+b}{2}$, resulting in the well-known mid-point inequality.

The Ostrowski result was further developed in [2], where Cerone and Dragomir presented a 3-point inequality and showed that the tightest bound is an average of the mid-point and trapezoidal rules. In the paper [1], Barnett and Dragomir developed a two dimensional version of the Ostrowski inequality.

In the current paper we combine the above two results and develop a two dimensional 3-point integral inequality for functions with bounded first derivatives. An application in the numerical integration of a two-dimensional integral is investigated.

3 The Results

Theorem 1. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a differentiable mapping on $[a_1, b_1] \times [a_2, b_2]$ and let $f''_{t_1, t_2} = \frac{\partial^2 f}{\partial t_1 \partial t_2}$ be bounded on $(a_1, b_1) \times (a_2, b_2)$. That is,

$$\|f''_{t_1, t_2}\|_\infty := \sup_{(x_1, x_2) \in (a_1, b_1) \times (a_2, b_2)} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right| < \infty.$$

Furthermore, let $x_i \in (a_i, b_i)$ and introduce the parameterization α_i, β_i defined by

$$\begin{aligned} \alpha_i &= (1 - \gamma_i) a_i + \gamma_i x_i, \\ \beta_i &= (1 - \gamma_i) b_i + \gamma_i x_i, \end{aligned}$$

where $\gamma_i \in [0, 1]$, for $i = 1, 2$. Then the following inequality holds

$$\begin{aligned} & \left| \sum_{k=1}^3 \sum_{j=1}^3 C_{k1} C_{j2} f_{jk} - \sum_{j=1}^3 (C_{j1} I_{j2} + C_{j2} I_{j1}) + \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1, t_2) dt_1 dt_2 \right| \\ & \leq \frac{\|f''_{t_1, t_2}\|_\infty}{4} (1 + (2\gamma_1 - 1)^2) \left[\left(\frac{b_1 - a_1}{2} \right)^2 + \left(x_1 - \frac{a_1 + b_1}{2} \right)^2 \right] \\ & \quad \times (1 + (2\gamma_2 - 1)^2) \left[\left(\frac{b_2 - a_2}{2} \right)^2 + \left(x_2 - \frac{a_2 + b_2}{2} \right)^2 \right], \end{aligned} \quad (3.1)$$

given that

$$(f_{jk}) = \begin{pmatrix} f(a_1, a_2) & f(x_1, a_2) & f(b_1, a_2) \\ f(a_1, x_2) & f(x_1, x_2) & f(b_1, x_2) \\ f(a_1, b_2) & f(x_1, b_2) & f(b_1, b_2) \end{pmatrix}, \quad (3.2)$$

$$(C_{jk}) = \begin{pmatrix} \gamma_1(x_1 - a_1) & \gamma_2(x_2 - a_2) \\ (1 - \gamma_1)(b_1 - a_1) & (1 - \gamma_2)(b_2 - a_2) \\ \gamma_1(b_1 - x_1) & \gamma_2(b_2 - a_2) \end{pmatrix}, \quad (3.3)$$

$$(I_{jk}) = \begin{pmatrix} \int_{a_1}^{b_1} f(t_1, a_2) dt_1 & \int_{a_2}^{b_2} f(a_1, t_2) dt_2 \\ \int_{a_1}^{b_1} f(t_1, x_2) dt_1 & \int_{a_1}^{b_1} f(x_1, t_2) dt_2 \\ \int_{a_1}^{b_1} f(t_1, b_2) dt_1 & \int_{a_1}^{b_1} f(b_1, t_2) dt_2 \end{pmatrix}. \quad (3.4)$$

Proof. Define the kernel

$$p(x, t) = \begin{cases} t - \alpha, & t \in [a, x], \\ t - \beta, & t \in (x, b], \end{cases} \quad (3.5)$$

where, as above, $\alpha = (1 - \gamma)a + \gamma x$, and $\beta = (1 - \gamma)b + \gamma x$. Using (3.5) and integrating by parts we obtain, after some simplification, the identity

$$\begin{aligned} & \int_a^b p(x, t) F'(t) dt \\ &= (1 - \gamma)(b - a)F(x) + \gamma[(x - a)F(a) + (b - x)F(b)] - \int_a^b F(t) dt. \end{aligned} \quad (3.6)$$

A two dimensional identity can be developed via repeated application of (3.6). To this end, we define the mapping

$$p_i(x_i, t_i) = \begin{cases} t_i - \alpha_i, & a_i \leq t_i \leq x_i, \\ t_i - \beta_i, & x_i < t_i \leq b_i, \end{cases} \quad \text{for } i = 1, 2. \quad (3.7)$$

Substituting p_1 for p and $f(t_1, \cdot)$ for $F(t)$ into (3.6) gives

$$\begin{aligned} \int_{a_1}^{b_1} p_1(x_1, t_1) \frac{\partial f}{\partial t_1} dt_1 &= (1 - \gamma_1)(b_1 - a_1)f(x_1, t_2) + \gamma_1(x_1 - a_1)f(a_1, t_2) \\ &+ \gamma_1(b_1 - x_1)f(b_1, t_2) - \int_{a_1}^{b_1} f(t_1, t_2) dt_1. \end{aligned} \quad (3.8)$$

Employing (3.6) again with p_2 as the kernel, $F(t_2) = \int_{a_1}^{b_1} p_1(x_1, t_1) \frac{\partial f}{\partial t_1} dt_1$ as the integrand and expanding with (3.8) produces the desired identity

$$\begin{aligned} \int_{a_2}^{b_2} p_2(x_2, t_2) F'(t_2) dt_2 &= \int_{a_2}^{b_2} \int_{a_1}^{b_1} p_2(x_2, t_2) p_1(x_1, t_1) \frac{\partial^2 f}{\partial t_1 \partial t_2} dt_1 dt_2 \\ &= (1 - \gamma_2)(b_2 - a_2)F(x_2) + \gamma_2 \left[(b_2 - x_2)F_2(b_2) + (x_2 - a_2)F_2(a_2) \right] \\ &\quad - \int_{a_2}^{b_2} F_2(t_2) dt_2 \\ &= \sum_{k=1}^3 \sum_{j=1}^3 C_{k1} C_{j2} f_{jk} - \sum_{j=1}^3 (C_{j1} I_{j2} + C_{j2} I_{j1}) + \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1, t_2) dt_1 dt_2. \end{aligned} \quad (3.9)$$

Assuming that both first partial derivatives of f are bounded, we can simply write down the inequality

$$\left| \int_{a_2}^{b_2} \int_{a_1}^{b_1} p_2(x_2, t_2) p_1(x_1, t_1) \frac{\partial^2 f}{\partial t_1 \partial t_2} dt_1 dt_2 \right| \leq \|f''_{t_1, t_2}\|_{\infty} \left(\int_{a_2}^{b_2} |p_2(x_2, t_2)| dt_2 \right) \left(\int_{a_1}^{b_1} |p_1(x_1, t_1)| dt_1 \right). \quad (3.10)$$

Now, consider

$$\begin{aligned} G_1(x_1) &= \int_{a_1}^{b_1} |p_1(x_1, t_1)| dt_1 \\ &= - \int_{a_1}^{\alpha_1} (t_1 - \alpha_1) dt_1 + \int_{\alpha_1}^{x_1} (t_1 - \alpha_1) dt_1 \\ &\quad - \int_{x_1}^{\beta_1} (t_1 - \beta_1) dt_1 + \int_{\beta_1}^{b_1} (t_1 - \beta_1) dt_1 \\ &= \frac{1}{2} [(\alpha_1 - a_1)^2 + (x_1 - \alpha_1)^2 + (\beta_1 - x_1)^2 + (b_1 - \beta_1)^2] \\ &= \frac{1}{2} [1 + (2\gamma_1 - 1)^2] \left[\left(\frac{b_1 - a_1}{2} \right)^2 + \left(x_1 - \frac{a_1 + b_1}{2} \right)^2 \right] \end{aligned} \quad (3.11)$$

Similarly, with $G_2(x_2) = \int_{a_2}^{b_2} |p_2(x_2, t_2)| dt_2$, we have

$$G_2(x) = \frac{1}{2} [1 + (2\gamma_2 - 1)^2] \left[\left(\frac{b_2 - a_2}{2} \right)^2 + \left(x_2 - \frac{a_2 + b_2}{2} \right)^2 \right]. \quad (3.12)$$

Using (3.2), (3.3) and (3.4) and substituting (3.9), (3.11) and (3.12) into (3.10) will produce the result (3.1) and thus the theorem is proved. \square

The following result gives an Ostrowski type inequality for double integrals. It involves double and single integrals together with a function evaluation at an interior point.

Corollary 1. *With the conditions as in Theorem 1, then*

$$\begin{aligned}
& \left| (b_1 - a_1)(b_2 - a_2)f(x_1, x_2) - (b_2 - a_2) \int_{a_1}^{b_1} f(t_1, x_2) dt_1 \right. \\
& \quad \left. - (b_1 - a_1) \int_{a_2}^{b_2} f(x_1, t_2) dt_2 + \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1, t_2) dt_1 dt_2 \right| \\
& \leq \|f''_{t_1, t_2}\|_{\infty} \left[\left(\frac{b_1 - a_1}{2} \right)^2 + \left(x_1 - \frac{a_1 + b_1}{2} \right)^2 \right] \left[\left(\frac{b_2 - a_2}{2} \right)^2 + \left(x_2 - \frac{a_2 + b_2}{2} \right)^2 \right].
\end{aligned} \tag{3.13}$$

Proof. Place $\gamma_1 = \gamma_2 = 0$ into equation (3.1). □

Thus, the earlier results of [1] and [8, p. 468] are reproduced as a special case of Theorem 1. We note that unlike [1], the proof for Theorem 1 can be easily extended to more than two dimensions.

Different values of the parameters γ_1 , γ_2 , x_1 and x_2 give rise to Newton-Cotes type inequalities for functions with bounded derivatives. For example $\gamma_1 = \gamma_2 = 0$, $x_1 = \frac{a_1 + b_1}{2}$ and $x_2 = \frac{a_2 + b_2}{2}$ produces the mid-point inequality; $\gamma_1 = \gamma_2 = 1$ a trapezoid-like inequality and $\gamma_1 = \gamma_2 = \frac{1}{3}$ a Simpson's like inequality.

From Theorem 1 it is a simple matter to show that the tightest bound is obtained when $\gamma_1 = \gamma_2 = \frac{1}{2}$ and x_1 and x_2 are at their mid-points. That is for the average of the mid-point and trapezoid inequalities.

Remark 1. *Let $f(t_1, t_2) = g(t_1)g(t_2)$ where $g : [a, b] \rightarrow \mathbb{R}$. If g is differentiable and satisfies the condition that $\|g'\|_{\infty} < \infty$, then, for $x_1 = x_2 = x$ and $\gamma_1 = \gamma_2 = \gamma$, we obtain a result from Theorem 1 which may be factored to recover the 3-point rule developed in [2], namely*

$$\begin{aligned}
& \left| \int_a^b g(t) dt - \gamma((x - a)g(a) + (b - x)g(b)) - (1 - \gamma)(b - a)g(x) \right| \\
& \leq \frac{\|g'\|_{\infty}}{2} (1 + (2\gamma - 1)^2) \left(\left(\frac{b - a}{2} \right)^2 + \left(x - \frac{a + b}{2} \right)^2 \right).
\end{aligned} \tag{3.14}$$

In general, cubature formulae are written only in terms of function evaluations, but Theorem 1 approximates a double integral in terms of single integrals and function evaluations. Therefore we write down the following

corollary which eliminates the one dimensional integrals by approximating them using the 3-point rule in equation (3.14). The resulting inequality has a coarser bound than equation (3.1).

Corollary 2. *Let f be given as in Theorem 1. Then*

$$\begin{aligned}
& \left| \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1, t_2) dt_1 dt_2 - \sum_{k=1}^3 \sum_{j=1}^3 C_{k1} C_{j2} f_{jk} \right| \\
& \leq \frac{\|f''_{t_1, t_2}\|_{\infty}}{4} (1 + (2\gamma_1 - 1)^2) (1 + (2\gamma_2 - 1)^2) \\
& \times \left[\left(\frac{b_1 - a_1}{2} \right)^2 + \left(x_1 - \frac{a_1 + b_1}{2} \right)^2 \right] \left[\left(\frac{b_2 - a_2}{2} \right)^2 + \left(x_2 - \frac{a_2 + b_2}{2} \right)^2 \right] \\
& \quad + \frac{1}{2} (1 + (2\gamma_1 - 1)^2) \left[\left(\frac{b_1 - a_1}{2} \right)^2 + \left(x_1 - \frac{a_1 + b_1}{2} \right)^2 \right] \\
& \times \left\{ \gamma_2 (x_2 - a_2) \|f'_{t_1, a_2}\|_{\infty} + (1 - \gamma_2) (b_2 - a_2) \|f'_{t_1, x_2}\|_{\infty} + \gamma_2 (b_2 - x_2) \|f'_{t_1, b_2}\|_{\infty} \right\} \\
& \quad + \frac{1}{2} (1 + (2\gamma_2 - 1)^2) \left[\left(\frac{b_2 - a_2}{2} \right)^2 + \left(x_2 - \frac{a_2 + b_2}{2} \right)^2 \right] \\
& \times \left\{ \gamma_1 (x_1 - a_1) \|f'_{a_1, t_2}\|_{\infty} + (1 - \gamma_1) (b_1 - a_1) \|f'_{x_1, t_2}\|_{\infty} + \gamma_1 (b_1 - x_1) \|f'_{b_1, t_2}\|_{\infty} \right\} \\
& \hspace{20em} (3.15)
\end{aligned}$$

Proof. Approximating each single integral in (3.1) by (3.14) and applying the triangle inequality produces the desired result. \square

Remark 2. *If $\gamma_1 = \gamma_2 = 0$ and $x_i = \frac{a_i + b_i}{2}$, then*

$$\begin{aligned}
& \left| \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1, t_2) dt_1 dt_2 - (b_1 - a_1) (b_2 - a_2) f\left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2}\right) \right| \\
& \leq \frac{\|f''_{t_1, t_2}\|_{\infty}}{16} (b_1 - a_1)^2 (b_2 - a_2)^2 + \frac{\|f'_{t_1, \frac{a_2 + b_2}{2}}\|_{\infty}}{4} (b_2 - a_2) (b_1 - a_1)^2 \\
& \quad + \frac{\|f'_{\frac{a_1 + b_1}{2}, t_2}\|_{\infty}}{4} (b_1 - a_1) (b_2 - a_2)^2. \quad (3.16)
\end{aligned}$$

4 Numerical Results

In this section the inequalities developed in Section 3 are used to approximate the double integral

$$\int_0^1 \int_0^1 1 - e^{-xy} dx dy = 0.203400400702947. \quad (4.1)$$

This integrand was chosen because integrating once in each direction is trivial. Namely, $\int_0^1 1 - e^{-xy} dx = \frac{y+e^{-y}-1}{y}$ and $\int_0^1 1 - e^{-xy} dy = \frac{x+e^{-x}-1}{x}$, but the double integral is not.

In Table 1, results are shown for the approximation to (4.1) using the rule and bound of (3.1). The numerical error is much smaller than the theoretical one and is smallest when Simpson’s rule is applied ($\gamma_1 = \gamma_2 = \frac{1}{3}$). The optimal theoretical bound is attained when $\gamma_1 = \gamma_2 = \frac{1}{2}$. It should be noted that $\gamma_1 = \gamma_2 = 0$ approximates (4.1) with the “mid-point” rule and employs one function evaluation (at the midpoint of the region) and two one-dimensional integrals (along the bi-sectors). The “trapezoidal” rule uses four sample points (the boundary corners) and four one-dimensional integrals (along the boundary). All other values, that is $\gamma_1, \gamma_2 \in (0, 1)$, produces a rule that is a linear combination of the above and results in the use of nine sample points and six one-dimensional integrals.

γ_1	γ_2	Numerical Error	Theoretical Error
0	0	1.5(-3)	6.3(-2)
$\frac{1}{3}$	$\frac{1}{3}$	5.4(-7)	1.9(-2)
0.5	0.5	4.3(-4)	1.6(-2)
1	1	6.5(-3)	6.3(-2)

Table 1: The numerical and theoretical errors in computing (4.1) using (3.1) with $x_1 = x_2 = 0.5$ and various values of γ_1, γ_2 .

To approximate (4.1) only in terms of function evaluations we use equation (3.15). The results are presented in Table 2. Since (3.15) is an approximation of (3.1), the results are qualitatively similar and quantitatively less accurate than those in Table 1. Simpson’s rule ($\gamma_1 = \gamma_2 = \frac{1}{3}$, nine sample points) is more accurate than the midpoint rule ($\gamma_1 = \gamma_2 = 0$, one sample point) which in turn is more accurate than the trapezoidal rule ($\gamma_1 = \gamma_2 = 1$, four sample points). We note that the theoretical errors are symmetric about $\gamma_1 = \gamma_2 = \frac{1}{2}$ in Table 1, but this is not the case in Table 2; these properties are easy to see by inspection of (3.1) and (3.15) respectively.

γ_1	γ_2	Numerical Error	Theoretical Error
0	0	1.8(-2)	4.6(-1)
$\frac{1}{3}$	$\frac{1}{3}$	9.3(-4)	2.2(-1)
0.5	0.5	1.0(-2)	1.9(-1)
1	1	4.5(-2)	3.8(-1)

Table 2: The numerical and theoretical errors in computing (4.1) using (3.15) with $x_1 = x_2 = 0.5$ and various values of γ_1, γ_2 .

To illustrate the use of a cubature formula, we form a composite rule from the inequality (3.13).

Theorem 2. *Let $f : [a_1, b_1] \times [a_2, b_2] \rightarrow \mathbb{R}$ be as in Theorem 1. Let I_n and J_m be arbitrary divisions of $[a_1, b_1]$ and $[a_2, b_2]$, respectively, i.e.*

$$I_n : a_1 = \xi_0 < \xi_1 < \dots < \xi_n = b_1, \quad \text{and} \quad J_m : a_2 = \tau_0 < \tau_1 < \dots < \tau_m = b_2.$$

Then we have the cubature formula

$$\begin{aligned} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1, t_2) dt_1 dt_2 &= \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i \nu_j f(x_i, y_j) - \sum_{i=0}^{n-1} h_i \int_{a_2}^{b_2} f(x_i, t_2) dt_2 \\ &\quad - \sum_{j=0}^{m-1} \nu_j \int_{a_1}^{b_1} f(t_1, y_j) dt_1 + R(f, I_n, J_m, \mathbf{x}, \mathbf{y}), \end{aligned} \quad (4.2)$$

where the remainder term R satisfies the condition

$$\begin{aligned} |R(f, I_n, J_m, \mathbf{x}, \mathbf{y})| &\leq \left\| f''_{t_1, t_2} \right\|_{\infty} \\ &\times \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[\left(\frac{h_i}{2} \right)^2 + \left(x_i - \frac{\xi_i + \xi_{i+1}}{2} \right)^2 \right] \left[\left(\frac{\nu_j}{2} \right)^2 + \left(y_j - \frac{\tau_j + \tau_{j+1}}{2} \right)^2 \right], \end{aligned} \quad (4.3)$$

and $h_i = \xi_{i+1} - \xi_i$, $\nu_j = \tau_{j+1} - \tau_j$, $x_i \in (\xi_i, \xi_{i+1})$, $y_j \in (\tau_j, \tau_{j+1})$ for $i = 0, 1, \dots, n-1$ and $j = 0, 1, \dots, m-1$.

Proof. Applying Corollary 1 on the interval $[\xi_i, \xi_{i+1}] \times [\tau_j, \tau_{j+1}]$, summing and using the triangle inequality produces the result. \square

Remark 3. If we were to use (4.2) to approximate the integral $\int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1, t_2) dt_1 dt_2$ with a uniform grid and sampling at each mid-point, then the remainder R is bounded by

$$|R(f, I_n, J_m, \mathbf{x}, \mathbf{y})| \leq \left\| f''_{t_1, t_2} \right\|_{\infty} \frac{(b_1 - a_1)^2 (b_2 - a_2)^2}{16nm} \quad (4.4)$$

n	m	Numerical Error	Error ratio	Theoretical Error
1	1	1.5(-3)	...	6.3(-2)
2	2	1.0(-4)	14.51	1.6(-2)
4	4	6.7(-6)	15.61	3.9(-3)
8	8	4.2(-7)	15.90	1.0(-3)
16	16	2.6(-8)	15.98	2.0(-4)
32	32	1.6(-9)	15.99	6.1(-5)
64	64	1.0 (-10)	16.00	1.5(-5)
128	128	6.6 (-12)	16.00	3.8(-6)

Table 3: The numerical and theoretical errors in evaluating (4.1) using the cubature rule in (4.2) for various values of n, m . Sampling occurs at the mid-point of each region.

Table 3 shows the numerical and theoretical errors in applying the mid-point cubature rule (4.2) to evaluate the double integral (4.1) for an increasing number of intervals. The numerical error ratio suggests that this composite rule has convergence

$$|R| \sim O\left(\frac{1}{n^2 m^2}\right).$$

This contrasts with (4.4) which predicts a convergence rate of

$$|R| \leq \frac{1}{16nm}.$$

It should be noted that the development of the bounds in Section 3 assumes that the integrand is once differentiable. This condition admits a wider class of functions than the usual bounds for Newton-Cotes rules, but the error estimate will be more conservative if its applied, as it is here, to an integrand that is infinitely smooth. In addition, theoretical optimality occurs at $\gamma_1 = \gamma_2 = \frac{1}{2}$, while numerically this value seems to be $\gamma_1 = \gamma_2 = \frac{1}{3}$. Because of the behaviour of the integrand, Simpson's rule which is optimal (in the Newton-Cotes sense) for the class of fourth differentiable mappings, will be superior. The methods of Section 3 can be applied to smoother [6] as well as weighted mappings [7, 9]. Work is continuing in this direction.

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