ON A NEW GENERALIZATION OF HILBERT'S INTEGRAL INEQUALITY AND ITS APPLICATIONS

B. YANG AND L. DEBNATH

ABSTRACT. This paper deals with a new generalization of Hilbert's integral inequality with a best possible constant factor involving the β function. As applications, general inequalities equivalent to Hilbert's type inequality and Hardy-Littlewood's integral inequality are proved.

1. INTRODUCTION

If $f, g \in L^2(0,\infty)$, $0 < \int_0^\infty f^2(x)\,dx < \infty$, and $0 < \int_0^\infty g^2(x)\,dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y}dxdy < \pi \left\{ \int_0^\infty f^2(x)\,dx \int_0^\infty g^2(x)\,dx \right\}^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is well known as Hilbert's inequality which has useful and important applications in mathematical analysis (see Hardy et al. [1]). In recent years, Hu [2] and Gao et al. [3] have made some important improvements of Hilbert's inequality (1.1). On the other hand, Yang [4] also generalized inequality (1.1) by introducing a parameter λ and proved the following:

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{(x+y)^\lambda}dxdy \\ &< B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) \left\{ \int_0^\infty x^{1-\lambda} f^2(x)\,dx \int_0^\infty x^{1-\lambda} g^2(x)\,dx \right\}^{\frac{1}{2}} \quad (0<\lambda\le 1), \end{aligned} \tag{1.2}$$

where $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ is a beta function involving the parameter λ. Similarly, Yang [5] also made a new generalization of the Hardy-Hilbert integral inequality.

The main objective of this paper is to make a new generalization of (1.2) by introducing two real parameters $\lambda\,(0<\lambda<\infty)$ and $\alpha \in \mathbb{R}$. We prove that the constant factor $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ in (1.2) and in its general version is the best possible. As applications, general inequalities equivalent to Hilbert's type inequality and Hardy-Hilbert's inequality are proved.

2. LEMMAS AND MAIN RESULTS

Lemma 1. *If $\lambda > 0$, $\alpha \in \mathbb{R}$, and $0 < \varepsilon < \frac{\lambda}{2}$, then*

$$\int_{\alpha+1}^\infty (x-\alpha)^{-1-\varepsilon} \left[\int_0^{\frac{1}{(x-\alpha)}} \frac{1}{(1+u)^\lambda} u^{\frac{(\lambda-2-\varepsilon)}{2}} du \right] dx = O(1) \qquad \left(\varepsilon \to 0^+\right) \tag{2.1}$$

Date: Setember 14, 1999.
1991 *Mathematics Subject Classification.* 26D15.
Key words and phrases. Hilbert's integral inequality, weight function and β function.

Proof. For $0 < \varepsilon < \frac{\lambda}{2}$, and $x \geq \alpha + 1$, we have

$$\int_0^{\frac{1}{(x-\alpha)}} \frac{1}{(1+u)^\lambda} u^{\frac{(\lambda-2-\varepsilon)}{2}} du < \int_0^{\frac{1}{(x-\alpha)}} u^{\frac{\left(\lambda-2-\frac{\lambda}{2}\right)}{2}} du = \frac{4}{\lambda}\left(\frac{1}{x-\alpha}\right)^{\frac{\lambda}{4}}.$$

Then, we obtain

$$\begin{aligned} 0 &< \int_{\alpha+1}^{\infty} (x-\alpha)^{-1-\varepsilon}\left[\int_0^{\frac{1}{(x-\alpha)}} \frac{1}{(1+u)^\lambda} u^{\frac{(\lambda-2-\varepsilon)}{2}} du\right] dx \\ &< \frac{4}{\lambda}\int_{\alpha+1}^{\infty} (x-\alpha)^{-1-\frac{\lambda}{4}} dx = \frac{16}{\lambda^2}. \end{aligned}$$

Thus, result (2.1) follows. This completes the proof.. ∎

Theorem 1. *If f and g are real functions, $\lambda > 0$, $0 < \int_\alpha^\infty (x-\alpha)^{1-\lambda} f^2(x)\,dx < \infty$, and $0 < \int_\alpha^\infty (x-\alpha)^{1-\lambda} g^2(x)\,dx < \infty$, then, for any $\alpha \in \mathbb{R}$, we have*

$$\begin{aligned} (2.2) \qquad & \int_\alpha^\infty \int_\alpha^\infty \frac{f(x)\,g(y)}{(x+y-2\alpha)^\lambda} dxdy \\ &< B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\int_\alpha^\infty (x-\alpha)^{1-\lambda} f^2(x)\,dx \int_\alpha^\infty (x-\alpha)^{1-\lambda} g^2(x)\,dx\right\}^{\frac{1}{2}}, \end{aligned}$$

where the constant factor $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) = \int_0^\infty \frac{1}{(1+u)^\lambda}\left(\frac{1}{u}\right)^{1-\frac{\lambda}{2}} du$ is the best possible.

In particular,

(i) *for $\lambda = 2n$ $(n \in \mathbb{N})$, we have*

$$\begin{aligned} (2.3) \qquad & \int_\alpha^\infty \int_\alpha^\infty \frac{f(x)\,g(y)}{(x+y-2\alpha)^{2n}} dxdy \\ &< \frac{[(n-1)!]^2}{(2n-1)!}\left\{\int_\alpha^\infty \frac{1}{(x-\alpha)^{2n-1}} f^2(x)\,dx \int_\alpha^\infty \frac{1}{(x-\alpha)^{2n-1}} g^2(x)\,dx\right\}^{\frac{1}{2}}; \end{aligned}$$

(ii) *for $\lambda = 2n+1$ $(n \in \mathbb{N})$, we have*

$$\begin{aligned} (2.4) \qquad & \int_\alpha^\infty \int_\alpha^\infty \frac{f(x)\,g(y)}{(x+y-2\alpha)^{2n+1}} dxdy \\ &< \frac{[(2n-1)!!]^2}{(4n)!!}\pi\left\{\int_\alpha^\infty \frac{1}{(x-\alpha)^{2n}} f^2(x)\,dx \int_\alpha^\infty \frac{1}{(x-\alpha)^{2n}} g^2(x)\,dx\right\}^{\frac{1}{2}}; \end{aligned}$$

(iii) *for $\lambda = 1$, we have*

$$(2.5) \qquad \int_\alpha^\infty \int_\alpha^\infty \frac{f(x)\,g(y)}{x+y-2\alpha} dxdy \leq \pi\left\{\int_\alpha^\infty f^2(x)\,dx \int_\alpha^\infty g^2(x)\,dx\right\}^{\frac{1}{2}},$$

where the constant factors in the above inequalities are still the best possible.

Proof. For $\lambda > 0$, $\alpha \in \mathbb{R}$, we define the weight function $w(x)$ by

$$(2.6) \qquad w(x) = \int_\alpha^\infty \frac{1}{(x+y-2\alpha)^\lambda}\left(\frac{x-\alpha}{y-\alpha}\right)^{1-\frac{\lambda}{2}} dy, \quad x \in (\alpha,\infty).$$

It follows from Cauchy's inequality that

$$
\begin{aligned}
(2.7) \qquad & \int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)\,g(y)}{(x+y-2\alpha)^{\lambda}}dxdy \\
= & \int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\left[\frac{f(x)}{(x+y-2\alpha)^{\frac{\lambda}{2}}}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{(2-\lambda)}{4}}\right] \\
& \times\left[\frac{g(y)}{(x+y-2\alpha)^{\frac{\lambda}{2}}}\left(\frac{y-\alpha}{x-\alpha}\right)^{\frac{(2-\lambda)}{4}}\right]dxdy \\
\leq & \left\{\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f^{2}(x)}{(x+y-2\alpha)^{\lambda}}\left(\frac{x-\alpha}{y-\alpha}\right)^{1-\frac{\lambda}{2}}dxdy\right. \\
& \left.\times\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{g^{2}(y)}{(x+y-2\alpha)^{\lambda}}\left(\frac{y-\alpha}{x-\alpha}\right)^{1-\frac{\lambda}{2}}dxdy\right\}^{\frac{1}{2}}.
\end{aligned}
$$

If (2.7) represents an equality for some f and g, then there exists constants c and d (see Kuang [6, p. 29]), such that

$$
\begin{aligned}
& c\frac{f^{2}(x)}{(x+y-2\alpha)^{\lambda}}\left(\frac{x-\alpha}{y-\alpha}\right)^{1-\frac{\lambda}{2}} \\
= \; & d\frac{g^{2}(y)}{(x+y-2\alpha)^{\lambda}}\left(\frac{y-\alpha}{x-\alpha}\right)^{1-\frac{\lambda}{2}} \quad \text{a.e in } (\alpha,\infty)\times(\alpha,\infty).
\end{aligned}
$$

Then we obtain $cf^{2}(x)(x-\alpha)^{2-\lambda} = dg^{2}(y)(y-\alpha)^{2-\lambda}$ a.e. in $(\alpha,\infty)\times(\alpha,\infty)$. Hence there exists constants a and b such that $ca = bd$, and

$$f^{2}(x) = a(x-\alpha)^{-2+\lambda} \text{ a.e in } (\alpha,\infty), \text{ and } g^{2}(y) = b(y-\alpha)^{-2+\lambda} \text{ a.e in } (\alpha,\infty).$$

This contradicts the following inequalities

$$0 < \int_{\alpha}^{\infty}(x-\alpha)^{-1-\lambda}f^{2}(x) < \infty, \quad \text{and} \quad 0 < \int_{\alpha}^{\infty}(x-\alpha)^{1-\lambda}g^{2}(x) < \infty.$$

It follows that (2.7) becomes a strict inequality.

Since (see Wang and Guo [7, p. 117])

$$\int_{0}^{\infty}\frac{1}{(1+u)^{\lambda}}\left(\frac{1}{u}\right)^{1-\frac{\lambda}{2}}du = \int_{0}^{\infty}\frac{u^{-1+\frac{\lambda}{2}}}{(1+u)^{\frac{\lambda}{2}+\frac{\lambda}{2}}}du = B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right),$$

putting $u = \frac{(y-\alpha)}{(x-\alpha)}$ in (2.6), we have

$$(2.8) \qquad w(x) = (x-\alpha)^{1-\lambda}\int_{0}^{\infty}\frac{1}{(1+u)^{\lambda}}\left(\frac{1}{u}\right)^{1-\frac{\lambda}{2}}du = (x-\alpha)^{1-\lambda}B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right).$$

Then, by (2.6), (2.7) and (2.8), we obtain

$$\begin{aligned}
&\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)\,g(y)}{(x+y-2\alpha)^{\lambda}}dxdy \\
< &\left\{\int_{\alpha}^{\infty} w(x)\, f^2(x)\,dx \int_{\alpha}^{\infty} w(y)\, g^2(y)\,dy\right\}^{\frac{1}{2}} \\
= &B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\int_{\alpha}^{\infty}(x-\alpha)^{1-\lambda} f^2(x)\,dx \int_{\alpha}^{\infty}(x-\alpha)^{1-\lambda} g^2(x)\,dx\right\}^{\frac{1}{2}}.
\end{aligned}$$

This shows that the inequality (2.2) is valid.

For any $\lambda > 0$, and $0 < \varepsilon < \frac{\lambda}{2}$, define $f_{\varepsilon}(t)$ as follows:

For $t \in (\alpha, \alpha+1)$, $f_{\varepsilon}(t) = 0$; and for $t \in [\alpha+1, \infty)$, $f_{\varepsilon}(t) = (t-\alpha)^{\frac{(\lambda-2-\varepsilon)}{2}}$.

Then, we obtain

$$\int_{\alpha}^{\infty}(x-\alpha)^{1-\lambda} f_{\varepsilon}^2(x)\,dx = \frac{1}{\varepsilon}.$$

Putting $u = \frac{(y-\alpha)}{(x-\alpha)}$, we have

$$\begin{aligned}
&\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f_{\varepsilon}(x)\,f_{\varepsilon}(y)}{(x+y-2\alpha)^{\lambda}}dxdy \qquad (2.9)\\
= &\int_{\alpha+1}^{\infty}(x-\alpha)^{\frac{(\lambda-2-\varepsilon)}{2}}\left[\int_{\alpha+1}^{\infty}\frac{1}{(x+y-2\alpha)^{\lambda}}(y-\alpha)^{\frac{(\lambda-2-\varepsilon)}{2}}dy\right]dx \\
= &\int_{\alpha+1}^{\infty}(x-\alpha)^{-1-\varepsilon}\left[\int_{\frac{1}{x-\alpha}}^{\infty}\frac{1}{(1+u)^{\lambda}}u^{\frac{(\lambda-2-\varepsilon)}{2}}du\right]dx \\
= &\int_{\alpha+1}^{\infty}(x-\alpha)^{-1-\varepsilon}\left[\int_{0}^{\infty}\frac{1}{(1+u)^{\lambda}}u^{\frac{(\lambda-2-\varepsilon)}{2}}du\right]dx - \\
&\int_{\alpha+1}^{\infty}(x-\alpha)^{-1-\varepsilon}\left[\int_{0}^{\frac{1}{x-\alpha}}\frac{1}{(1+u)^{\lambda}}u^{\frac{(\lambda-2-\varepsilon)}{2}}du\right]dx \\
= &\frac{1}{\varepsilon}\int_{0}^{\infty}\frac{1}{(1+u)^{\lambda}}\left(\frac{1}{u}\right)^{\frac{(2-\lambda+\varepsilon)}{2}}du - O(1) \qquad (\text{by } (2.1)) \\
= &\frac{1}{\varepsilon}\left[\int_{0}^{\infty}\frac{1}{(1+u)^{\lambda}}\left(\frac{1}{u}\right)^{1-\frac{\lambda}{2}}du + o(1)\right] - O(1) \\
= &\frac{1}{\varepsilon}\left(B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) + o(1)\right) \quad \left(\varepsilon \to 0^{+}\right).
\end{aligned}$$

Suppose there exists a number $\lambda > 0$ such that the constant $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ in (2.2) is not the best possible. Then there exists a positive number $K_{\lambda}\left(< B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right)$ such that

$$\begin{aligned}
&\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f_{\varepsilon}(x)\,f_{\varepsilon}(y)}{(x+y-2\alpha)^{\lambda}}dxdy \\
< &K_{\lambda}\left\{\int_{\alpha}^{\infty}(x-\alpha)^{1-\lambda} f^2(x)\,dx \int_{\alpha}^{\infty}(x-\alpha)^{1-\lambda} g^2(x)\,dx\right\}^{\frac{1}{2}}.
\end{aligned}$$

In particular, we have

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f_{\varepsilon}(x)\,f_{\varepsilon}(y)}{(x+y-2\alpha)^{\lambda}}dxdy<K_{\lambda}\int_{\alpha}^{\infty}(x-\alpha)^{1-\lambda}f_{\varepsilon}^{2}(x)\,dx=\frac{1}{\varepsilon}K_{\lambda}. \tag{2.10}$$

Making $\varepsilon\,(>0)$ as small as possible such that, in (2.9), that makes $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)+o(1)>K_{\lambda}$, then we have

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f_{\varepsilon}(x)\,f_{\varepsilon}(y)}{(x+y-2\alpha)^{\lambda}}dxdy>\frac{1}{\varepsilon}K_{\lambda}.$$

This contradicts (2.10). Hence for any $\lambda>0$, the constant factor $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ in (2.2) is the best possible.

Since

$$B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)=\frac{\left[\Gamma\left(\frac{\lambda}{2}\right)\right]^{2}}{\Gamma(\lambda)},\ \Gamma(n)=(n-1)!,\ \Gamma\left(n+\frac{1}{2}\right)=\frac{(2n-1)!!}{2^{n}}\sqrt{\pi}\quad(n\in\mathbb{N}),$$

and $B\left(\frac{1}{2},\frac{1}{2}\right)=\pi$, by (2.2), we have (2.3), (2.4) and (2.5).

Thus the proof of the theorem is complete. ∎

Remark 1. *If $\alpha=0$, (2.5) reduces to (1.1), and (2.2) reduces to (1.2). In view of the fact that $\lambda\in(0,\infty)$, we have new inequalities (2.3) and (2.4) with the possible best constant factors. This shows that inequality (2.2) is a new generalization of (1.1).*

Remark 2. *If $\alpha=-\frac{1}{2}$, inequality (2.2) corresponds to a new inequality involving the double series proved by Yang and Debnath* [8]*:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_{n}b_{m}}{(m+n+1)^{\lambda}} < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}a_{n}^{2}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}b_{n}^{2}\right\}^{\frac{1}{2}}\quad(0<\lambda\leq 2). \tag{2.11}$$

3. Applications

Theorem 2. *If f is a real function, $\lambda>0$, and $0<\int_{\alpha}^{\infty}(x-\alpha)^{1-\lambda}f^{2}(x)\,dx<\infty$, then, for any $\alpha\in\mathbb{R}$, we have*

$$\int_{\alpha}^{\infty}(y-\alpha)^{\lambda-1}\left[\int_{\alpha}^{\infty}\frac{f(x)}{(x+y-2\alpha)^{\lambda}}dx\right]^{2}dy < \left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right]^{2}\int_{\alpha}^{\infty}(x-\alpha)^{1-\lambda}f^{2}(x)\,dx, \tag{3.1}$$

where the constant factor $\left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right]^{2}$ is the best possible. Inequality (3.1) is equivalent to (2.2).

In particular, we obtain the following:

(i) *for* $\lambda = 2n\ (n \in \mathbb{N})$, *we have*

$$\int_{\alpha}^{\infty} (y-\alpha)^{2n-1} \left[\int_{\alpha}^{\infty} \frac{f(x)}{(x+y-2\alpha)^{2n}} dx\right]^2 dy < \left\{\frac{[(n-1)!]^2}{(2n-1)!}\right\}^2 \int_{\alpha}^{\infty} \frac{1}{(x-\alpha)^{2n-1}} f^2(x)\, dx; \tag{3.2}$$

(ii) *for* $\lambda = 2n+1\ (n \in \mathbb{N})$, *we find*

$$\int_{\alpha}^{\infty} (y-\alpha)^{2n} \left[\int_{\alpha}^{\infty} \frac{f(x)}{(x+y-2\alpha)^{2n+1}} dx\right]^2 dy < \left\{\frac{[(2n-1)!!]^2}{(4n)!!}\pi\right\}^2 \int_{\alpha}^{\infty} \frac{1}{(x-\alpha)^{2n}} f^2(x)\, dx; \tag{3.3}$$

(iii) *for* $\lambda = 1$, *we have*

$$\int_{\alpha}^{\infty} \left(\int_{\alpha}^{\infty} \frac{f(x)}{x+y-2\alpha} dx\right)^2 dy < \pi^2 \int_{\alpha}^{\infty} f^2(x)\, dx. \tag{3.4}$$

Proof. Since $0 < \int_{\alpha}^{\infty} (x-\alpha)^{1-\lambda} f^2(x)\, dx < \infty$, then there exists a constant $T_0\ (> \alpha)$ such that for any $T > T_0$, we have

$$0 < \int_{\alpha}^{T} (x-\alpha)^{1-\lambda} f^2(x)\, dx < \infty.$$

Define $g(T, y)$ by

$$g(T, y) = (y-\alpha)^{\lambda-1} \int_{\alpha}^{T} \frac{|f(x)|}{(x+y-2\alpha)^{\lambda}} dx,\ y \in (\alpha, T] \quad (T > T_0).$$

It follows from (2.2) that

$$\begin{aligned} 0 &< \left[\int_{\alpha}^{T} (y-\alpha)^{1-\lambda} g^2(T, y)\, dy\right]^2 \\ &= \left\{\int_{\alpha}^{T} (y-\alpha)^{\lambda-1} \left[\int_{\alpha}^{T} \frac{|f(x)|}{(x+y-2\alpha)^{\lambda}} dx\right]^2 dy\right\}^2 \\ &= \left[\int_{\alpha}^{T}\int_{\alpha}^{T} \frac{|f(x)|\, g(T, y)}{(x+y-2\alpha)^{\lambda}} dx dy\right]^2 \\ &< \left[B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\right]^2 \int_{\alpha}^{T} (x-\alpha)^{1-\lambda} f^2(x)\, dx \int_{\alpha}^{T} (x-\alpha)^{1-\lambda} g^2(T, x)\, dx. \end{aligned} \tag{3.5}$$

Thus, we find

$$\begin{aligned}(3.6)\qquad & \int_{\alpha}^{T}(y-\alpha)^{\lambda-1}\left[\int_{\alpha}^{T}\frac{f(x)}{(x+y-2\alpha)^{\lambda}}dx\right]^{2}dy \\ \leq & \int_{\alpha}^{T}(y-\alpha)^{1-\lambda}g^{2}(T,y)\,dy \\ = & \int_{\alpha}^{T}(y-\alpha)^{\lambda-1}\left[\int_{\alpha}^{T}\frac{|f(x)|}{(x+y-2\alpha)^{\lambda}}dx\right]^{2}dy \\ < & \left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right]^{2}\int_{\alpha}^{T}(x-\alpha)^{1-\lambda}f^{2}(x)\,dx.\end{aligned}$$

It follows that $0<\int_{\alpha}^{\infty}(y-\alpha)^{1-\lambda}g^{2}(\infty,y)\,dx<\infty$. Then, by (2.2), when $T\to\infty$, (3.5) is still a strict inequality; and so too for inequality (3.6). It follows that inequality (3.1) is valid.

If inequality (3.1) holds, then by Cauchy's inequality, we have

$$\begin{aligned}(3.7)\qquad & \int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)\,g(y)}{(x+y-2\alpha)^{\lambda}}dxdy \\ = & \int_{\alpha}^{\infty}\left\{y^{\frac{(\lambda-1)}{2}}\int_{\alpha}^{\infty}\frac{f(x)}{(x+y-2\alpha)^{\lambda}}dx\right\}\left[y^{\frac{(1-\lambda)}{2}}g(y)\right]dy \\ \leq & \int_{\alpha}^{\infty}\left\{(y-\alpha)^{\lambda-1}\left[\int_{\alpha}^{\infty}\frac{f(x)}{(x+y-2\alpha)^{\lambda}}dx\right]^{2}dy\int_{\alpha}^{\infty}(y-\alpha)^{1-\lambda}g^{2}(y)\,dy\right\}^{\frac{1}{2}}.\end{aligned}$$

In view of (3.1), we find (2.2). This means that inequality (3.1) is equivalent to (2.2).

If there exists a number $\lambda>0$ such that the constant $\left\lfloor B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right\rfloor^{2}$ in (3.1) is not the best possible, then, by (3.7), we show that, for this $\lambda>0$, the constant $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ in (2.2) is not the best possible. This is a contradiction. It is obvious that inequalities (3.2), (3.4) and (3.5) are valid.

This proves the theorem. ∎

Remark 3. *When $\alpha=0$, (3.1) reduces to*

$$\begin{aligned}(3.8)\qquad & \int_{0}^{\infty}y^{\lambda-1}\left[\int_{0}^{\infty}\frac{f(x)}{(x+y)^{\lambda}}dx\right]^{2}dy \\ < & \left[B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\right]^{2}\int_{0}^{\infty}x^{1-\lambda}f^{2}(x)\,dx\quad(\lambda\in(0,\infty)).\end{aligned}$$

The parameter $\lambda>0$, in (3.8) has a larger range than inequality (12) stated by Yang [4].

Let $f\in L^{2}(0,1)$, $0<\int_{0}^{1}f^{2}(x)\,dx<\infty$, and $a_{n}=\int_{0}^{1}x^{n}f(x)\,dx$ $(n=0,1,2,...)$. The well known Hardy-Littlewood inequality has the following form:

$$(3.9)\qquad \sum_{n=0}^{\infty}a_{n}^{2}<\pi\int_{0}^{1}f^{2}(x)\,dx,$$

where the constant π is the best possible (see Hardy et al. [1, Chap. 9]). Recently, Gao [9] gave an integral form of (3.9) as follows:

Let $h \in L^2(0,1)$, $0 < \int_0^1 h^2(x)\,dx < \infty$, and $f(x) = \int_0^1 t^x h(t)\,dt$ $(x \in [0,\infty))$. Then

$$\int_0^\infty f^2(x)\,dx < \pi \int_0^1 h^2(t)\,dt. \tag{3.10}$$

In the following theorem, we give a new generalization of (3.10) by using (2.3).

Theorem 3. *If $\alpha \in \mathbb{R}$, $h \in L^2(0,1)$, $0 < \int_0^1 h^2(x)\,dx < \infty$, and $f_\alpha(x) = \int_0^1 t^{x-\alpha-\frac{1}{2}} h(t)\,dt$, $x \in (\alpha,\infty)$, then*

$$\int_\alpha^\infty f_\alpha^2(x)\,dx < \pi \int_0^1 h^2(t)\,dt. \tag{3.11}$$

Proof. Since $f_\alpha^2(x) > 0$, for any $[a,b] \subset (\alpha,\infty)$, we have $0 < \int_a^b f_\alpha^2(x)\,d < \infty$. By (2.5), we still have

$$\int_a^b \int_a^b \frac{f_\alpha(x) f_\alpha(y)}{x+y-2u} dxdy < \pi \int_a^b f_\alpha^2(x)\,dx.$$

Then, by Cauchy's inequality, we obtain

$$\begin{aligned}
&\left(\int_a^b f_\alpha^2(x)\,dx\right)^2 \\
&= \left(\int_a^b f_\alpha(x) \int_0^1 t^{x-\alpha-\frac{1}{2}} h(t)\,dtdx\right)^2 \\
&= \left\{\int_0^1 \left(\int_a^b f_\alpha(x)\, t^{x-\alpha-\frac{1}{2}} dx\right) h(t)\,dt\right\}^2 \\
&\le \int_0^1 \left(\int_a^b f_\alpha(x)\, t^{x-\alpha-\frac{1}{2}} dx\right)^2 dt \int_0^1 h^2(t)\,dt \\
&= \int_0^1 \int_a^b \int_a^b f_\alpha(x) f_\alpha(y)\, t^{x+y-2\alpha-1} dxdydt \int_0^1 h^2(t)\,dt \\
&= \int_a^b \int_a^b f_\alpha(x) f_\alpha(y) \left\{\int_0^1 t^{x+y-2\alpha-1} dt\right\} dxdy \int_0^1 h^2(t)\,dt \\
&= \int_a^b \int_a^b \frac{f_\alpha(x) f_\alpha(y)}{(x+y-2\alpha)} dxdy \int_0^1 h^2(t)\,dt \\
&< \pi \int_a^b f_\alpha^2(x)\,dx \int_0^1 h^2(t)\,dt.
\end{aligned} \tag{3.12}$$

Thus, we have

$$0 < \int_a^b f_\alpha^2(x)\,dx < \pi \int_0^1 h^2(t)\,dt, \quad \text{and} \quad 0 < \int_\alpha^\infty f_\alpha^2(x)\,dx \le \pi \int_0^1 h^2(t)\,dt < \infty.$$

Hence, setting $a \to \alpha$, $b \to \infty$ in (3.12), we still have a strict inequality. Inequality (3.11) holds. Thus, the theorem is proved. ∎

Remark 4. *When* $\alpha = -\frac{1}{2}$, *setting* $f(x) = f_{-\frac{1}{2}}(x) = \int_0^1 t^x h(t)\, dt \ \left(x \in \left(-\frac{1}{2}, \infty\right)\right)$, *inequality (3.11) reduces to*

$$\int_{-\frac{1}{2}}^{\infty} f^2(x)\, dx < \pi \int_0^1 h^2(t)\, dt. \tag{3.13}$$

Since $\int_0^\infty f^2(x)\, dx < \int_{-\frac{1}{2}}^1 f^2(x)\, dx$, *it follows that (3.13) is a corresponding Hardy-Littlewood integral inequality, which gives an improvement of (3.10). Hence, inequality (3.11) is a new generalization of (3.13).*

References

[1] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, London, 1952.

[2] Hu Ke, On Hilbert's inequality, *Chinese Ann. Math.,* **Ser B13**, (1992), 35-39.

[3] Gao Mingzhe, Tan Li and L. Debnath, Some improvements on Hilbert's integral inequality, *J. Math. Anal. & Appl.,* **229** (1999), 682-689.

[4] Yang Bicheng, On Hilbert's integral inequality, *J. Math. Anal. & Appl.,* **220** (1998), 778-785.

[5] Yang Bicheng, On Generalizations of Hardy-Hilbert's integral inequality, *Acta Mathematica Sinica,* **41** (1998), 839-844.

[6] Kuang Jichang, *Applied Inequalities,* Hunan Education Press, Changsha, 1993.

[7] Wang Zhuxi and Guo Dunren, *Introduction to particular functions,* Science Press, Beijing, 1979.

[8] Yang Bicheng and L. Debnath, On a new generalization of Hardy-Hilbert's inequality and its applications, *J. Math. Anal. & Appl.,* **233** (1999), 484-497.

[9] Gao Mingzhe, On Hilbert's inequality and its applications, *J. Math. Anal. & Appl.,* **212** (1997), 316-323.

Department of Mathematics, Guangdong Education College, Guangzhou, Guangdong 510303, The People's Republic of China

Department of Mathematics, University of Central Florida, Orlando, Florida 32816, U.S.A.

E-mail address: `ldebnath@pegasus.cc.ucf.edu`