SOME RESULTS ON L^1- APPROXIMATION OF THE $r-$TH DERIVATE OF FOURIER SERIES

Z. TOMOVSKI

Abstract. In this paper we obtain the criterions for L^1-convergence of the $r-$th derivatives of the cosine and sine trigonometric series. An equivalent form of the condition S_r, r =0,1,2, ... considered in [4] is given and also an extension of Sidon's theorem [2] is made. For the $r-$th derivate of the sine series $g(x) = \sum_{n=1}^{\infty} a_n \sin nx$, $(a_n \in S_r,\ r = 0, 1, 2, ...)$ the following estimate is proved

$$\int_{\frac{\pi}{m+1}}^{\pi} \left|g^{(r)}(x)\right| dx = \sum_{n=1}^{m} |a_n| \cdot n^{r-1} + O_r\left(\sum_{n=1}^{\infty} n^r A_n\right),$$

where $m = 1, 2, 3, ...$ and O_r depends only on r.

1. Introduction

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1.1}$$

$$g(x) = \sum_{n=1}^{\infty} a_n \sin nx \tag{1.2}$$

be the well-known cosine and sine trigonometric series.

In [2] Sidon proved the following theorem.

Theorem 1. *(Sidon). Let $\{a_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ be sequences such that $|\alpha_n| \le 1$, for every n and let $\sum_{n=1}^{\infty} |p_n|$ be convergent. If*

$$a_n = \sum_{k=n}^{\infty} \frac{p_k}{k} \sum_{l=n}^{k} \alpha_l\,, n \in \mathbb{N}\,, \tag{1.3}$$

then the cosine series (1.1) is the Fourier series of its sum f.

Several authors have studied the problem of L^1-convergence of the series (1.1) and (1.2).

In [3] Telyakovskii defined the following class of L^1-convergence of Fourier series. A sequence $\{a_k\}$ belongs to the class S, or $\{a_k\} \in S$ if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} A_k < \infty$ and $|\Delta a_k| \le A_k$ for all k.

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The importance of Telyakovskii's contributions are twofold. Firstly, he expressed Sidon's conditions (1.3) in a succinct equivalent form, and secondly, he showed that the class S is also a class of L^1- convergence. Thus the class S is usually called the Sidon-Telyakovskii class.

In the same paper, Telyakovskii proved the following two theorems.

Theorem 2. [3]. *Let the coefficients of the series $f(x)$ satisfy the condition S. Then the series is a Fourier series and the following relation holds:*

$$\int_0^{\pi} |f(x)|\,dx \le M \sum_{n=1}^{\infty} A_n,\ M > 0.$$

Theorem 3. [3]. *Let the coefficients of the series $g(x)$ satisfy the condition S. Then the following relation holds for $p = 1, 2, 3, \ldots$*

$$\int_{\frac{\pi}{p+1}}^{\pi} |g(x)|\,dx = \sum_{n=1}^{p} \frac{|a_n|}{n} + O\left(\sum_{n=1}^{\infty} A_n\right)$$

In particular, $g(x)$ is a Fourier series iff $\sum_{k=1}^{\infty} \frac{|a_n|}{n} < \infty$.

In [4] we extended the Sidon-Telyakovskii class, i.e., we defined the class S_r, $r = 0, 1, 2, 3, \ldots$ as follows: $\{a_k\} \in S_r$ if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} k^r A_k < \infty$ and $|\Delta a_k| \le A_k$ for all k.

We note that by $A_k \downarrow 0$ and $\sum_{k=1}^{\infty} k^r A_k < \infty$, we get

$$k^{r+1} A_k = o(1),\ k \to \infty. \tag{1.4}$$

When $r = 0$, denote $S_r = S$. It is obvious that $S_{r+1} \subset S_r$ but the converse of that inclusion is false.

Example 1. *For $n = 1, 2, 3, \ldots$ define $a_n = \frac{1}{n}$ and $A_n = \frac{1}{n^2+1}$.*
Since $\Delta a_n = \frac{1}{n(n+1)}$, $|\Delta a_n| \le A_n = \frac{1}{n^2+1}$ and $A_n \downarrow 0$, we have:

$$\sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \frac{1}{n^2+1} < \infty \ \text{ i.e. } \{a_n\} \in S.$$

But the series $\sum_{n=1}^{\infty} nA_n = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is divergent, i.e. $\{a_n\} \notin S_1$.

In the same paper [4] we proved the following theorem.

Theorem 4. [4] *Let the coefficients of the series (1.1) satisfy the condition S_r, $r = 0, 1, 2, \ldots$. Then the $r-$th derivative of the series (1.1) is a Fourier series of some $f^{(r)} \in L^1(0, \pi)$ and the following relation holds:*

$$\int_0^{\pi} \left|f^{(r)}(x)\right| dx \le M \sum_{n=1}^{\infty} n^r A_n,\ M > 0.$$

This is an extension of the Telyakovskii Theorem 2.

2. Results

In this paper, we proved the following main results.

Theorem 5. *Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ be sequences such that $|\alpha_n| \le 1$, for every n and let $\sum_{n=1}^{\infty} n^r |p_n| < \infty$, $r = 0, 1, 2, \ldots$. If*

$$a_n = \sum_{k=n}^{\infty} \frac{p_k}{k} \sum_{l=n}^{k} \alpha_l, \; n \in \mathbb{N}, \tag{2.1}$$

then the $r-$th derivative of the series (1.1) is a Fourier series of some $f^{(r)} \in L^1$.

Theorem 6. *Let the coefficients of the series $g(x)$ satisfy the condition S_r, $r = 0, 1, 2, \ldots$. Then the following relation holds for $m = 1, 2, 3, \ldots$*

$$\int_{\frac{\pi}{m+1}}^{\pi} \left| g^{(r)}(x) \right| dx = \sum_{n=1}^{m} |a_n| \cdot n^{r-1} + O_r\left(\sum_{n=1}^{\infty} n^r A_n \right),$$

where O_r depends only on r.

In particular, $g^{(r)}(x)$ is a Fourier series iff $\sum_{n=1}^{\infty} |a_n| \cdot n^{r-1} < \infty$.

3. Lemmas

For the proof of our new theorems we need the following lemmas.

Lemma 1. [1] *Let r be a nonnegative integer and $x \in (0, \pi]$, where $n \ge 1$. Then*

$$\begin{aligned} D_n^{(r)}(x) &= \sum_{k=0}^{r-1} \frac{\left(n+\frac{1}{2}\right)^k \sin\left[\left(n+\frac{1}{2}\right)x + \frac{k\pi}{2}\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-k}} \varphi_k(x) \\ &\quad + \frac{\left(n+\frac{1}{2}\right)^r \sin\left[\left(n+\frac{1}{2}\right)x + \frac{r\pi}{2}\right]}{2\sin\left(\frac{x}{2}\right)}, \end{aligned}$$

where the same φ_k denotes various analytical functions of x, independent of n.

Lemma 2. *Let $\{\alpha_j\}_{j=0}^{k}$ be a sequence of real numbers. Then the following relation holds for $\nu = 0, 1, 2, \ldots, r$ and $r = 0, 1, 2, \ldots$*

$$\begin{aligned} U_k &= \int_{\frac{\pi}{k+1}}^{\pi} \left| \sum_{j=0}^{k} \alpha_j \frac{\left(j+\frac{1}{2}\right)^\nu \sin\left[\left(j+\frac{1}{2}\right)x + \frac{\nu+3}{2}\pi\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-\nu}} \right| dx \\ &= O_r\left((k+1)^{r-\nu+\frac{1}{2}} \left(\sum_{j=0}^{k} \alpha_j^2 (j+1)^{2\nu} \right)^{\frac{1}{2}} \right), \end{aligned}$$

where O_r depends only on r.

Proof. Applying the Cauchy-Bunjakovskii inequality yields

$$\begin{aligned} U_k &\le \left[\int_{\frac{\pi}{k+1}}^{\pi} \frac{dx}{\left(\sin\left(\frac{x}{2}\right)\right)^{2(r+1-\nu)}} \right]^{\frac{1}{2}} \\ &\quad \times \left\{ \int_{\frac{\pi}{k+1}}^{\pi} \left[\sum_{j=0}^{k} \alpha_j \left(j+\frac{1}{2}\right)^\nu \sin\left[\left(j+\frac{1}{2}\right)x + \frac{\nu+3}{2}\pi\right] \right]^2 dx \right\}^{\frac{1}{2}}. \end{aligned}$$

Since

$$\int_{\frac{\pi}{k+1}}^{\pi} \frac{dx}{\left(\sin\left(\frac{x}{2}\right)\right)^{2(r+1-\nu)}} \le \pi^{2(r+1-\nu)} \int_{\frac{\pi}{k+1}}^{\pi} \frac{dx}{x^{2(r+1-\nu)}} \le \frac{\pi (k+1)^{2(r+1-\nu)-1}}{2(r+1-\nu)-1},$$

we have

$$\begin{aligned} U_k &\le \left[\frac{\pi}{2(r+1-\nu)-1}\right]^{\frac{1}{2}} \left[(k+1)^{2(r+1-\nu)-1}\right]^{\frac{1}{2}} \\ &\times \left\{\int_0^{\pi} \left[\sum_{j=0}^{k} \alpha_j \left(j+\frac{1}{2}\right)^{\nu} \sin\left[\left(j+\frac{1}{2}\right)x + \frac{\nu+3}{2}\pi\right]\right]^2 dx\right\}^{\frac{1}{2}} \\ &= \left[\frac{2\pi}{2(r+1-\nu)-1}\right]^{\frac{1}{2}} \left[(k+1)^{2(r+1-\nu)-1}\right]^{\frac{1}{2}} \\ &\times \left\{\int_0^{2\pi} \left[\sum_{j=0}^{k} \alpha_j \left(j+\frac{1}{2}\right)^{\nu} \sin\left[(2j+1)t + \frac{\nu+3}{2}\pi\right]\right]^2 dt\right\}^{\frac{1}{2}}. \end{aligned}$$

Then, applying Parseval's equality, we obtain:

$$U_k \le \left[\frac{2\pi}{2(r+1-\nu)-1}\right]^{\frac{1}{2}} \left[(k+1)^{2(r+1-\nu)-1}\right]^{\frac{1}{2}} \left[\sum_{j=0}^{k} |\alpha_j|^2 \left(j+\frac{1}{2}\right)^{2\nu}\right]^{\frac{1}{2}}.$$

Finally,

$$U_k = O_r\left((k+1)^{r-\nu+\frac{1}{2}} \left(\sum_{j=0}^{k} \alpha_j^2 (j+1)^{2\nu}\right)^{\frac{1}{2}}\right).$$

∎

Lemma 3. *Let $\{a_k\}$ be a sequence of real numbers such that $|\alpha_k| \le 1$, for all k. Then there exists a constant $M_r > 0$ such that for any $n \ge 1$ and $r = 0, 1, 2, 3, ...$*

$$\int_{\frac{\pi}{n+1}}^{\pi} \left|\sum_{k=0}^{n} \alpha_k \bar{D}_k^{(r)}(x)\right| dx \le M_r (n+1)^{r+1}.$$

Proof. Since $-\cos\left(n+\frac{1}{2}\right)x = \sin\left[\left(n+\frac{1}{2}\right)x + \frac{3\pi}{2}\right]$, by Lemma 1, we get:

$$\begin{aligned} \bar{D}_n^{(r)}(x) &= \sum_{k=0}^{r-1} \frac{\left(n+\frac{1}{2}\right)^k \sin\left[\left(n+\frac{1}{2}\right)x + \frac{k+3}{2}\pi\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-k}} \varphi_k(x) \\ &+ \frac{\left(n+\frac{1}{2}\right)^r \sin\left[\left(n+\frac{1}{2}\right)x + \frac{r+3}{2}\pi\right]}{2\sin\left(\frac{x}{2}\right)}, \end{aligned}$$

where the same φ_k denotes various analytical functions of x, independent of n. Now, we have:

$$\begin{aligned}
&\int_{\frac{\pi}{n+1}}^{\pi}\left|\sum_{k=0}^{n}\alpha_k\bar{D}_k^{(r)}(x)\right|dx \\
\leq\ &\int_{\frac{\pi}{n+1}}^{\pi}\left|\sum_{j=0}^{n}\alpha_j\left(\sum_{\nu=0}^{r-1}\frac{\left(j+\frac{1}{2}\right)^{\nu}\sin\left[\left(j+\frac{1}{2}\right)x+\frac{\nu+3}{2}\pi\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-\nu}}\varphi_\nu(x)\right)\right|dx \\
&+\int_{\frac{\pi}{n+1}}^{\pi}\left|\sum_{j=0}^{n}\alpha_j\frac{\left(j+\frac{1}{2}\right)^{r}\sin\left[\left(j+\frac{1}{2}\right)x+\frac{r+3}{2}\pi\right]}{2\sin\left(\frac{x}{2}\right)}\varphi_\nu(x)\right|dx \\
=\ &\lambda_n+\mu_n
\end{aligned}$$

Since φ_ν are bounded, we have:

$$\int_{\frac{\pi}{n+1}}^{\pi}\left|\sum_{j=0}^{n}\alpha_j\frac{\left(j+\frac{1}{2}\right)^{\nu}\sin\left[\left(j+\frac{1}{2}\right)x+\frac{\nu+3}{2}\pi\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-\nu}}\varphi_\nu\right|dx\leq K_rU_n\,,$$

where U_n is the integral as in Lemma 2, and K_r is a positive constant dependent on r.

Applying Lemma 2 to the last integral, we obtain:

$$\begin{aligned}
&\int_{\frac{\pi}{n+1}}^{\pi}\left|\sum_{j=0}^{n}\alpha_j\frac{\left(j+\frac{1}{2}\right)^{\nu}\sin\left[\left(j+\frac{1}{2}\right)x+\frac{\nu+3}{2}\pi\right]}{\left(\sin\left(\frac{x}{2}\right)\right)^{r+1-\nu}}\varphi_\nu(x)\right|dx \\
=\ &O_r\left((n+1)^{r+\frac{1}{2}-\nu}\left(\sum_{j=0}^{n}\alpha_j^2(j+1)^{2\nu}\right)^{\frac{1}{2}}\right) \\
=\ &O_r\left((n+1)^{r+\frac{1}{2}-\nu}(n+1)^{\nu+\frac{1}{2}}\right)=O_r\left((n+1)^{r+1}\right)\,,
\end{aligned}$$

where O_r depends only on r. Since r is a finite value, we have:

$$\lambda_n=O_r\left((n+1)^{r+1}\right)\,.$$

Similarly, we can get:

$$\mu_n=O_r\left((n+1)^{r+1}\right)\,.$$

Finally our inequality is satisfied, where M_r is an absolute constant depending only on r. ▮

Remark 1. *For* $r=0$, *we obtain the Telyakovskii inequality, proved in* [3].

4. Proofs of the Main Results

4.1. **Proof of Theorem 5.** By Theorem 4, it suffices to show that the condition S_r, $r=0,1,2,3,...$ is equivalent with the condition (2.1).

Let (2.1) hold. Then

$$\Delta a_k\leq\alpha_k\sum_{m=k}^{\infty}\frac{p_m}{m}$$

and we denote

$$A_k = \sum_{m=k}^{\infty} \frac{|p_m|}{m}.$$

Since $|a_k| \le 1$, we get

$$|\Delta a_k| \le |\alpha_k| \sum_{m=k}^{\infty} \frac{|p_m|}{m} \le A_k\,, \text{ for all } k.$$

However,

$$\sum_{k=1}^{\infty} k^r A_k = \sum_{k=1}^{\infty} k^r \sum_{m=k}^{\infty} \frac{|p_m|}{m} \le \sum_{k=1}^{\infty} k^r \sum_{m=k}^{\infty} |p_m| \le \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} m^r |p_m| < \infty\,,$$

and $A_k \downarrow 0$ i.e. $\{a_k\} \in S_r$.

Now, if $\{a_k\} \in S_r$, we put $\alpha_k = \frac{\Delta a_k}{A_k}$ and $p_k = k\,(A_k - A_{k+1})$. Hence $|\alpha_k| \le 1$, and by (1.4) we get:

$$\sum_{k=1}^{\infty} k^r |p_k| = \sum_{k=1}^{\infty} k^{r+1} (A_k - A_{k+1}) = \sum_{k=1}^{\infty} k^r A_k < \infty\,.$$

Finally,

$$a_k = \sum_{i=k}^{\infty} \Delta a_k = \sum_{i=k}^{\infty} \alpha_i A_i = \sum_{i=k}^{\infty} \alpha_i \sum_{m=i}^{\infty} \Delta A_m = \sum_{i=k}^{\infty} \alpha_i \sum_{m=i}^{\infty} \frac{p_m}{m} = \sum_{m=k}^{\infty} \frac{p_m}{m} \sum_{i=k}^{m} \alpha_i\,,$$

i.e. (2.1) holds.

4.2. **Proof of Theorem 6.** We suppose that $a_0 = 0$ and $A_0 = \max(|a_1|, A_1)$.

For $r = 0$, applying Abel's transformation, we have:

$$g(x) = \sum_{k=0}^{\infty} \Delta a_k \bar{D}_k(x)\,. \tag{4.1}$$

From inequality

$$\left|\bar{D}_n^{(r)}(x)\right| = O\left(\frac{n^r}{x}\right),\ x \in \left[\frac{\pi}{m+1}, \pi\right],\ m = 1, 2, 3... \tag{4.2}$$

we have that the series $\sum_{k=0}^{\infty} \Delta a_k \bar{D}_k^{(r)}(x)$ is uniformly convergent on any compact subset of $x \in \left[\frac{\pi}{m+1}, \pi\right]$, $m = 1, 2, 3, ...$

Thus, representation (4.1) implies that

$$g^{(r)}(x) = \sum_{k=0}^{\infty} \Delta a_k \bar{D}_k^{(r)}(x)\,.$$

Then,

$$\begin{aligned}
&\int_{\frac{\pi}{m+1}}^{\pi}\left|g^{(r)}(x)\right|dx\\
=\;&\int_{\frac{\pi}{m+1}}^{\pi}\left|\sum_{k=0}^{\infty}\Delta a_k\bar{D}_k^{(r)}(x)\right|dx\\
=\;&\sum_{j=1}^{m}\int_{\frac{\pi}{j+1}}^{\frac{\pi}{j}}\left|\sum_{k=0}^{j-1}\Delta a_k\bar{D}_k^{(r)}(x)\right|dx+O\left(\sum_{j=1}^{m}\int_{\frac{\pi}{j+1}}^{\frac{\pi}{j}}\left|\sum_{k=j}^{\infty}\Delta a_k\bar{D}_k^{(r)}(x)\right|dx\right).
\end{aligned}$$

Let

$$I_1=\sum_{j=1}^{m}\int_{\frac{\pi}{j+1}}^{\frac{\pi}{j}}\left|\sum_{k=0}^{j-1}\Delta a_k\bar{D}_k^{(r)}(x)\right|dx\,,\ I_2=\sum_{j=1}^{m}\int_{\frac{\pi}{j+1}}^{\frac{\pi}{j}}\left|\sum_{k=j}^{\infty}\Delta a_k\bar{D}_k^{(r)}(x)\right|dx\,.$$

Applying the inequality (4.2) we have:

$$\begin{aligned}
I_1 &= O\left(\sum_{j=1}^{m}\int_{\frac{\pi}{j+1}}^{\frac{\pi}{j}}\left|\sum_{k=0}^{j-1}\Delta a_k\right|j^r\frac{dx}{x}\right)\\
&= O\left(\sum_{j=1}^{m}|a_j|\,j^r\ln\left(1+\frac{1}{j}\right)\right)=O\left(\sum_{j=1}^{m}|a_j|\,j^{r-1}\right).
\end{aligned}$$

Application of Abel's transformation yields

$$\sum_{k=j}^{\infty}\Delta a_k\bar{D}_k^{(r)}(x)=\sum_{k=j}^{\infty}\Delta A_k\sum_{i=0}^{k}\frac{\Delta a_i}{A_i}\bar{D}_i^{(r)}(x)-A_j\sum_{i=0}^{j-1}\frac{\Delta a_i}{A_i}\bar{D}_i^{(r)}(x)\,.$$

Let us estimate the second integral:

$$I_2\le\sum_{j=1}^{m}\left[\sum_{k=j}^{\infty}(\Delta A_k)\int_{\frac{\pi}{j+1}}^{\frac{\pi}{j}}\left|\sum_{i=0}^{k}\frac{|\Delta a_i|}{A_i}\bar{D}_i^{(r)}(x)\right|dx+A_j\int_{\frac{\pi}{j+1}}^{\frac{\pi}{j}}\left|\sum_{i=0}^{j-1}\frac{|\Delta a_i|}{A_i}\bar{D}_i^{(r)}(x)\right|dx\right].$$

Applying Lemma 3, we have

$$J_k=\int_{\frac{\pi}{j+1}}^{\frac{\pi}{j}}\left|\sum_{i=0}^{k}\frac{|\Delta a_i|}{A_i}\bar{D}_i^{(r)}(x)\right|dx=O_r\left((k+1)^{r+1}\right).\tag{4.3}$$

However,

$$I_j=\int_{\frac{\pi}{j+1}}^{\frac{\pi}{j}}\left|\sum_{i=0}^{j}\frac{|\Delta a_i|}{A_i}\bar{D}_i^{(r)}(x)\right|dx=O\left(\int_{\frac{\pi}{j+1}}^{\frac{\pi}{j}}j^r\left(\sum_{i=0}^{j}\frac{|\Delta a_i|}{A_i}\right)\frac{dx}{x}\right)=O\left(j^r\right)\tag{4.4}$$

and finally by (4.3), (4.4) and (1.4), we have

$$\begin{aligned} I_2 &\leq \sum_{k=1}^{\infty} (\Delta A_k) \cdot J_k + O\left(\sum_{j=1}^{\infty} j^r A_j\right) \\ &= O_r(1) \sum_{k=1}^{\infty} (\Delta A_k)(k+1)^{r+1} + O\left(\sum_{j=1}^{\infty} j^r A_j\right) \\ &= O_r\left(\sum_{j=1}^{\infty} j^r A_j\right). \end{aligned}$$

References

[1] Sheng Shuyun, *The extensions of the theorem of C.V. Stanojevic and V.B. Stanojevic,* Proc. Amer. Math. Soc., **110** (1990), 895-904.

[2] S. Sidon, *Hinrehichende Bedingungen fur den Fourier-Charakter einer trigonometrischen Reihe,* J. London Math. Soc., **72** (1939), 158-160.

[3] S.A. Telyakovskii, *On a sufficient condition of Sidon for the integrability of trigonometric series,* Math. Notes, **14** (1973), 742-748.

[4] Z. Tomovski, *An extension of the Sidon-Fomin inequality and applications,* Analysis Mathematica (to appear).

Faculty of Mathematical and Natural Sciences, Skopje, Macedonia
E-mail address: tomovski@iunona.pmf.ukim.edu.mk