ERROR BOUNDS FOR THE PERTURBATION OF THE DRAZIN INVERSE OF CLOSED OPERATORS WITH EQUAL SPECTRAL PROJECTIONS

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Abstract. We study perturbations of the Drazin inverse of a closed linear operator A for the case when the perturbed operator has the same spectral projection as A. This theory subsumes results recently obtained by Wei and Wang, Rakočević and Wei, and Castro and Koliha. We give explicit error estimates for the perturbation of Drazin inverse, and error estimates involving higher powers of the operators.

1. Introduction

In several recent papers [3, 4, 14, 17, 18], perturbations of the Drazin inverse were studied with a purpose to obtain explicit error bounds. In this paper we present a perturbation theory for the Drazin inverse A^{D} of a closed linear operator A in which the perturbed operator B shares the spectral projection at 0 with A.

By $\mathcal{C}(X)$ we denote the set of all closed linear operators acting on a linear subspace of X to X, where X is a complex Banach space. We write $\mathcal{D}(A)$, $\mathcal{N}(A)$, $\mathcal{R}(A)$ and $\sigma(A)$ for the domain, nullspace, range and spectrum of an operator $A \in \mathcal{C}(X)$. All relevant concepts from theory of closed linear operators can be found in [16]. The set of all operators $T \in \mathcal{C}(X)$ with $\mathcal{D}(T) = X$ will be denoted by $\mathcal{B}(X)$; we recall that operators in $\mathcal{B}(X)$ are bounded, and the operator norm of $T \in \mathcal{B}(X)$ will be denoted by $\|T\|$.

An operator $A \in \mathcal{C}(X)$ is *quasipolar* if 0 is not an accumulation point of the spectrum of A. We recall the following result of [11], which can be deduced from [16, Theorem V.9.2].

Lemma 1.1. *An operator $A \in \mathcal{C}(X)$ is quasipolar if and only if there exists a projection $P \in \mathcal{B}(X)$ such that*

(i) $\mathcal{R}(P) \subset \mathcal{D}(A)$,

(ii) $PAx = APx$ *for all* $x \in \mathcal{D}(A)$,

(iii) $A + P \in \mathcal{C}(X)$ *is invertible,*

(iv) $AP \in \mathcal{B}(X)$ *is quasinilpotent, that is,* $\sigma(AP) = \{0\}$.

The projection P is uniquely determined by conditions (i)–(iv)*, and it is the spectral projection of A at* 0.

Definition 1.2. An operator $A \in \mathcal{C}(X)$ is *Drazin invertible* if it can be expressed in the form (relative to a topological direct sum $X = X_1 \oplus X_2$)

$$A = A_1 \oplus A_2, \text{ where } A_1 \text{ is closed invertible and } A_2 \text{ is bounded quasinilpotent.} \tag{1.1}$$

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The *Drazin index* $i(A)$ of A is 0 if A is invertible; if A is not invertible, $i(A)$ is the least positive integer k for which $A_2^k = 0$, or ∞ if no such integer k exists. The operators

$$A^{\mathrm{D}} = A_1^{-1} \oplus 0 \quad \text{and} \quad A^\pi = 0 \oplus I \tag{1.2}$$

are the *Drazin inverse* of A and the *spectral projection* of A corresponding to 0, respectively.

This definition given in [11] generalizes the concept of pseudoinverse introduced by Drazin [6] in two directions. It applies to closed linear operators, whereas [6] can be applied only to bounded linear operators (elements of an algebra), and it admits an infinite index in the case when A_2 is a true quasinilpotent, while only a finite index was possible in [6].

In the preceding definition, $A^{\mathrm{D}} \in \mathcal{B}(X)$ and $A^\pi \in \mathcal{B}(X)$. The following equations, easily verifiable by a manipulation of direct operator sums, are often useful:

$$A^\pi = I - AA^{\mathrm{D}}, \qquad A^{\mathrm{D}} = (A + A^\pi)^{-1}(I - A^\pi). \tag{1.3}$$

Drazin invertible operators include invertible and quasinilpotent operators when $X_2 = \{0\}$ and $X_1 = \{0\}$, respectively. From Lemma 1.1 and Definition 1.2 we obtain the following result.

Lemma 1.3. *$A \in \mathcal{C}(X)$ is Drazin invertible if and only if A is quasipolar.*

Mbekhta [12] proved that the spaces X_1 and X_2 in the direct sum $X = X_1 \oplus X_2$ inducing (1.1) are $X_1 = \mathcal{K}(A)$ and $X_2 = \mathcal{H}_0(A)$, where

$$\begin{aligned}\mathcal{H}_0(A) &= \{x \in \mathcal{D}_\infty(A) : \limsup_{n\to\infty} \|A^n x\|^{1/n} = 0\},\\ \mathcal{K}(A) &= \{x \in X : \exists x_n \in \mathcal{D}_n(A) \text{ such that}\\ &\qquad Ax_1 = x,\ Ax_{n+1} = x_n \text{ for } n = 1, 2, \dots\\ &\qquad \text{and } \limsup_{n\to\infty} \|x_n\|^{1/n} < \infty\}.\end{aligned}$$

They are hyperinvariant under A, and

$$\mathcal{N}(A^n) \subset \mathcal{H}_0(A), \quad \mathcal{K}(A) \subset \mathcal{R}(A^n), \quad n = 1, 2, \dots$$

It is known [9] that $A \in \mathcal{C}(X)$ is quasipolar if and only if $X = \mathcal{K}(A) \oplus \mathcal{H}_0(A)$, where at least one of the spaces $\mathcal{K}(A)$ and $\mathcal{H}_0(A)$ is closed. For a Drazin invertible operator $A \in \mathcal{C}(X)$ we have

$$\mathcal{R}(A^{\mathrm{D}}) = \mathcal{D}(A) \cap \mathcal{K}(A), \qquad \mathcal{N}(A^{\mathrm{D}}) = \mathcal{H}_0(A).$$

2. Characterizing operators that satisfy $B^\pi = A^\pi$

Let $A \in \mathcal{C}(X)$ be a Drazin invertible operator. Our first task is to characterize those Drazin invertible operators $B \in \mathcal{C}(X)$, with $\mathcal{D}(B) = \mathcal{D}(A)$, for which $B^\pi = A^\pi$.

One result that will be used systematically throughout is that the product ST of $S \in \mathcal{C}(X)$ and of $T \in \mathcal{B}(X)$ with $\mathcal{R}(T) \subset \mathcal{D}(S)$ is in $\mathcal{B}(X)$.

Let $A \in \mathcal{C}(X)$ be a Drazin invertible operator, and let $B \in \mathcal{C}(X)$ satisfy $\mathcal{D}(B) = \mathcal{D}(A)$. We have

$$\mathcal{D}(A) = \mathcal{R}(A^{\mathrm{D}}) + \mathcal{R}(A^\pi),$$

which means that the operator products BA^π and BA^{D} are well defined (and in $\mathcal{B}(X)$). Relative to the space decomposition $X = \mathcal{K}(A) \oplus \mathcal{H}_0(A)$ we have the operator matrices

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad A^\pi = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \quad A^{\mathrm{D}} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

We focus our attention on two conditions on B that will be used in our main result. From the matrix representations of operators we can deduce that

$$A^\pi BA^{\rm D} = 0 \Leftrightarrow B_{21} = 0 \Leftrightarrow B(\mathcal{D}(A)\cap\mathcal{K}(A)) \subset \mathcal{K}(A),$$
$$A^{\rm D}BA^\pi = 0 \Leftrightarrow B_{12} = 0 \Leftrightarrow B(\mathcal{H}_0(A)) \subset \mathcal{H}_0(A).$$

These conditions are fulfilled automatically for $B = A$.

Before proceeding, we mention a principle that will be used in the sequel without a further comment. If $U = B - A$ is the difference of two closed operators with the same domain, then U is a linear operator with the domain $\mathcal{D}(A)$, not necessarily closed. However, $UT \in \mathcal{B}(X)$ for any operator $T \in \mathcal{B}(X)$ with $\mathcal{R}(T) \subset \mathcal{D}(A)$ since $UT = AT - BT$.

We now give a characterization of operators $B \in \mathcal{C}(X)$ with $\mathcal{D}(B) = \mathcal{D}(A)$ that satisfy $B^\pi = A^\pi$.

Theorem 2.1. *Let $A \in \mathcal{C}(X)$ be a Drazin invertible operator, and let $B \in \mathcal{C}(X)$ satisfy $\mathcal{D}(B) = \mathcal{D}(A)$. Then the following are equivalent*:

(i) *B is Drazin invertible and $B^\pi = A^\pi$*;

(ii) *B is Drazin invertible, $A^\pi BA^{\rm D} = 0$, $I + (B - A)A^{\rm D}$ is invertible and*

$$B^{\rm D} = A^{\rm D}(I + (B - A)A^{\rm D})^{-1}; \tag{2.1}$$

(iii) *$A^\pi BA^{\rm D} = 0 = A^{\rm D}BA^\pi$, $I + (B - A)A^{\rm D}$ is invertible and BA^π is quasinilpotent*;

(iv) *$A^\pi BA^{\rm D} = 0 = A^{\rm D}BA^\pi$, BA^π is quasinilpotent and $B + A^\pi$ is invertible.*

Proof. Throughout this proof we write

$$C = I + (B - A)A^{\rm D} \ (\in \mathcal{B}(X)).$$

(i) $\Rightarrow$ (ii) Since $B^\pi = A^\pi$, relative to the space decomposition $X = \mathcal{K}(A) \oplus \mathcal{H}_0(A)$ we have

$$A = A_1 \oplus A_2, \qquad B = B_1 \oplus B_2,$$

where A_1, B_1 are invertible with $\mathcal{D}(A_1) = \mathcal{D}(B_1) = \mathcal{K}(A)\cap\mathcal{D}(A)$, and A_2, B_2 are quasinilpotent with $\mathcal{D}(A_2) = \mathcal{D}(B_2) = \mathcal{H}_0(A)$. Then

$$C = I \oplus I + (B_1 - A_1)A_1^{-1} \oplus 0 = B_1A_1^{-1} \oplus I. \tag{2.2}$$

Hence $C \in \mathcal{B}(X)$ is invertible with $C^{-1} = A_1B_1^{-1} \oplus I$, and

$$A^{\rm D}C^{-1} = (A_1^{-1} \oplus 0)(A_1B_1^{-1} \oplus I) = B_1^{-1} \oplus 0 = B^{\rm D}.$$

Moreover, $A^\pi BA^{\rm D} = (0 \oplus I)(B_1A_1^{-1} \oplus 0) = 0$.

(ii) $\Rightarrow$ (i) First we observe that $A^\pi C = A^\pi + A^\pi(B - A)A^{\rm D} = A^\pi$. Hence

$$\begin{aligned} A^\pi - B^\pi &= BB^{\rm D} - AA^{\rm D} = BA^{\rm D}C^{-1} - I + A^\pi \\ &= (BA^{\rm D} - C + A^\pi C)C^{-1} = (BA^{\rm D} - C + A^\pi)C^{-1} \\ &= (BA^{\rm D} - BA^{\rm D})C^{-1} = 0. \end{aligned}$$

(i) $\Rightarrow$ (iii) Suppose that (i) holds. In view of Lemma 1.1, BA^π is quasinilpotent, and $A^\pi Bx = BA^\pi x$ for all $x \in \mathcal{D}(A)$, which implies $A^\pi BA^{\rm D} = 0 = A^{\rm D}BA^\pi$. The invertibility of C follows from the first part of this proof.

(iii) $\Rightarrow$ (iv) Condition $A^\pi BA^{\rm D} = 0 = A^{\rm D}BA^\pi$ is equivalent to $A^\pi Bx = BA^\pi x$ for all $x \in \mathcal{D}(A)$; hence we have $B = B_1 \oplus B_2$ relative to the space decomposition $X = \mathcal{K}(A)\oplus\mathcal{H}_0(A)$. By (2.2), $C = B_1A_1^{-1} \oplus I$; since C is invertible, so is B_1. Further, $BA^\pi = 0 \oplus B_2$ is quasinilpotent, which implies that B_2 is also quasinilpotent. Hence $B + A^\pi = B_1 \oplus (I + B_2)$ is invertible.

(iv) ⇒ (i) Follows from Lemma 1.1. □

It is of some interest to observe that condition $A^{\mathrm{D}}BA^{\pi} = 0$ alone in (iv) would still ensure that B is Drazin invertible, but not that A^{π} is the spectral projection of B at 0; for that we need the mirror condition $A^{\pi}BA^{\mathrm{D}} = 0$. In the case that the operators A and B are in $\mathcal{B}(X)$, we obtain the following specialization of the preceding theorem in which condition $A^{\pi}BA^{\mathrm{D}} = 0$ can be omitted from (ii). We note that for bounded operators $A^{\pi}BA^{\mathrm{D}} = 0 = A^{\mathrm{D}}BA^{\pi}$ is equivalent to $A^{\pi}A = AA^{\pi}$.

Corollary 2.2. *Let $A \in \mathcal{B}(X)$ be a Drazin invertible operator, and let $B \in \mathcal{B}(X)$. Then the following are equivalent*:

(i) *B is Drazin invertible and $B^{\pi} = A^{\pi}$*;

(ii) *B is Drazin invertible, $I + (B - A)A^{\mathrm{D}}$ is invertible and* (2.1) *holds*;

(iii) *$A^{\pi}B = BA^{\pi}$, $I + (B - A)A^{\mathrm{D}}$ is invertible and BA^{π} is quasinilpotent*;

(iv) *$A^{\pi}B = BA^{\pi}$, BA^{π} is quasinilpotent and $B + A^{\pi}$ is invertible.*

Proof. The only part that requires proof is (ii) ⇒ (i). First we note that for operators in $\mathcal{B}(X)$, $\mathcal{R}(A^{\mathrm{D}}) = \mathcal{K}(A)$ and $\mathcal{N}(A^{\mathrm{D}}) = \mathcal{H}_0(A)$. Hence the result will follow when we show that

$$\mathcal{R}(A^{\mathrm{D}}) = \mathcal{R}(B^{\mathrm{D}}) \quad \text{and} \quad \mathcal{N}(A^{\mathrm{D}}) = \mathcal{N}(B^{\mathrm{D}}).$$

But this clearly follows from

$$B^{\mathrm{D}} = A^{\mathrm{D}}(I + UA^{\mathrm{D}})^{-1} = (I + A^{\mathrm{D}}U)^{-1}A^{\mathrm{D}},$$

where $U = B - A$; $I + A^{\mathrm{D}}U$ is invertible since $\sigma(A^{\mathrm{D}}U) \setminus \{0\} = \sigma(UA^{\mathrm{D}}) \setminus \{0\}$. □

The preceding corollary generalizes [5, Theorem 2.1] proved for matrices.

3. Perturbations of the Drazin inverse in the case $B^{\pi} = A^{\pi}$

The continuity properties of the Drazin inverse for matrices are well known [1, 2]; the continuity of the conventional Drazin inverse for bounded linear operators was investigated by Rakočević in [13], and the continuity of the generalized Drazin inverse for bounded operators by Koliha and Rakočević in [10]. It was proved in [10] that if A and B_{α} are bounded linear operators such that $B_{\alpha} \to A$, then

$$B_{\alpha}^{\mathrm{D}} \to A^{\mathrm{D}} \Leftrightarrow B_{\alpha}^{\pi} \to A^{\pi}$$

in the operator norm. However, no explicit bounds for the perturbations of the Drazin inverse have been obtained for this general case. Most of the previous studies of error bounds limit themselves to special cases of perturbations satisfying $B^{\pi} = A^{\pi}$. Let A be a Drazin invertible operator (for the moment we may assume that A is bounded), that $A = A_1 \oplus A_2$ with A_1 invertible and A_2 quasinilpotent, and that $B = A + U$, where $U \in \mathcal{B}(X)$ commutes with A^{π}. Wei [17], Wei and Wang [18], and Rakočević and Wei [14] study perturbations of the type

$$B = (A_1 + U_1) \oplus A_2,$$

while Castro and Koliha [3] and Castro, Koliha and Straškraba [4] investigate perturbations

$$B = (A_1 + U_1) \oplus (A_2 + U_2),$$

where U_2 is quasinilpotent (or nilpotent in the case of matrices) and commutes with A_2. The results of [3, 14, 17, 18] include explicit error bounds for the Drazin inverse and relations between the Drazin indices of B and A.

In this section we address general perturbations of the Drazin inverse of a closed linear operator A under the assumption that the perturbed operator B satisfies the condition $B^\pi = A^\pi$. We consider a closed, rather than just a bounded, operator because important applications of perturbation theory are to infinitesimal generators of operator C_0-semigroups, and such operators are closed.

We now give our main perturbation theorem for the Drazin inverse of closed linear operators.

THEOREM 3.1. *Let $A \in \mathcal{C}(X)$ be a Drazin invertible operator, let $B \in \mathcal{C}(X)$ with $\mathcal{D}(B) = \mathcal{D}(A)$ and let $U = B - A$. If*

$$\|UA^{\rm D}\| < 1, \quad A^\pi BA^{\rm D} = 0 = A^{\rm D}BA^\pi, \quad \sigma(BA^\pi) = \{0\}, \tag{3.1}$$

then B is a Drazin invertible operator, and

$$\mathcal{K}(B) = \mathcal{K}(A), \quad \mathcal{H}_0(B) = \mathcal{H}_0(A), \tag{3.2}$$

$$B^{\rm D} = A^{\rm D}(I + UA^{\rm D})^{-1}, \tag{3.3}$$

$$\frac{\|B^{\rm D} - A^{\rm D}\|}{\|A^{\rm D}\|} \le \frac{\|UA^{\rm D}\|}{1 - \|UA^{\rm D}\|}, \tag{3.4}$$

$$\frac{\|A^{\rm D}\|}{1 + \|UA^{\rm D}\|} \le \|B^{\rm D}\| \le \frac{\|A^{\rm D}\|}{1 - \|UA^{\rm D}\|}. \tag{3.5}$$

If A, $B \in \mathcal{B}(X)$, (3.4) can be improved to

$$\frac{\|UA^{\rm D}\|}{\kappa_{\rm D}(A)(1 + \|A^{\rm D}\|\|U\|)} \le \frac{\|B^{\rm D} - A^{\rm D}\|}{\|A^{\rm D}\|} \le \frac{\|UA^{\rm D}\|}{1 - \|UA^{\rm D}\|} \le \frac{\kappa_{\rm D}(A)\|U\|/\|A\|}{1 - \kappa_{\rm D}(A)\|U\|/\|A\|}, \tag{3.6}$$

where $\kappa_{\rm D}(A) = \|A\|\|A^{\rm D}\|$ is the Drazin condition number of A.

Proof. From $\theta = \|UA^{\rm D}\| < 1$ we deduce that $I + UA^{\rm D}$ is invertible. Then condition (iii) of Theorem 2.1 is fulfilled, and B is Drazin invertible with $B^\pi = A^\pi$. Since $\mathcal{R}(A^\pi) = \mathcal{H}_0(A)$ and $\mathcal{N}(A^\pi) = \mathcal{K}(A)$, we obtain (3.2). Formula (3.3) for the Drazin inverse of B follows from Theorem 2.1. To prove (3.4), we first observe that $\|(I + UA^{\rm D})^{-1}\| = \|\sum_{n=0}^\infty (-1)^n (UA^{\rm D})^n\| \le \sum_{n=0}^\infty \theta^n = (1-\theta)^{-1}$, which in view of (3.3) implies

$$\|B^{\rm D} - A^{\rm D}\| = \|A^{\rm D}UA^{\rm D}(I + UA^{\rm D})^{-1}\| \le \|A^{\rm D}\|\theta(1-\theta)^{-1}.$$

Inequality (3.5) follows similarly from (3.3).

Let A, $B \in \mathcal{B}(X)$. Since $B^\pi = A^\pi$, $AA^{\rm D} = BB^{\rm D}$ and $UA^{\rm D} = (A + U)(A^{\rm D} - B^{\rm D})$. From the last expression we deduce the lower estimate for $\|B^{\rm D} - A^{\rm D}\|/\|A^{\rm D}\|$ in (3.6) taking into account that $\kappa_{\rm D}(A) \ge \|I - A^\pi\| \ge 1$. □

COROLLARY 3.2. *Let $A \in \mathcal{C}(X)$ be a Drazin invertible operator and let (B_n) be a sequence of operators in $\mathcal{C}(X)$ with $\mathcal{D}(B_n) = \mathcal{D}(A)$ such that*

$$A^\pi B_n A^{\rm D} = 0 = A^{\rm D} B_n A^\pi, \quad \sigma(B_n A^\pi) = \{0\}, \quad n = 1, 2, \dots$$

and

$$\varepsilon_n = \|(B_n - A)A^{\rm D}\| \to 0 \quad \textit{as } n \to \infty.$$

For all sufficiently large n the operators B_n are Drazin invertible with $B_n^\pi = A^\pi$, and

$$\|B_n^{\rm D} - A^{\rm D}\| \le \frac{\|A^{\rm D}\|\,\varepsilon_n}{1 - \varepsilon_n} \to 0 \quad \textit{as } n \to \infty.$$

In the following two remarks we make comparison of our results on perturbation of the Drazin inverse with existing literature. To this end we consider only a restricted type of perturbations of A, where $A \in \mathcal{C}(X)$ is Drazin invertible and $B = A + U$ for some $U \in \mathcal{B}(X)$.

Remark 3.3. Condition

$$\|A^{\mathrm{D}}U\| < 1, \quad A^\pi U = 0 = UA^\pi \tag{3.7}$$

is a special case of (3.1) since $(A+U)A^\pi = AA^\pi$ is quasinilpotent by Lemma 1.1; (3.7) is equivalent to the condition $(\mathcal{W})$ that was used in [17, 18] for matrices and in [14] for bounded linear operators and elements of Banach algebras. Hence we recover the perturbation results of [18] and [14] as a special case of Theorem 3.1.

Remark 3.4. The preceding theorem subsumes the perturbation results of [3, 4], where A and U satisfied the condition

$$\|A^{\mathrm{D}}U\| < 1, \quad A^\pi U = UA^\pi, \quad AA^\pi U = UAA^\pi, \quad UA^\pi \text{ is quasinilpotent,} \tag{3.8}$$

which is a special case of (3.1). Indeed,

$$(A+U)A^\pi = AA^\pi + UA^\pi$$

is quasinilpotent being the sum of two commuting quasinilpotent operators in $\mathcal{B}(X)$.

The results of [3, 14, 18] include relations between the Drazin indices of A and B. Under condition (3.7) adopted in [14, 18] we have $i(B) = i(A)$. If A and U satisfy condition (3.8) as in [3], then

$$|i(A) - i(A^\pi U)| + 1 \le i(B) \le i(A) + i(A^\pi U) - 1.$$

If only (3.1) is assumed, no relation between $i(A)$ and $i(B)$ exists. In fact, in the next example we show that for any pair of extended natural numbers $p, q \in \mathbb{N} \cup \{\infty\}$, we can find a pair of operators A and B satisfying (3.1) such that $i(A) = p$ and $i(B) = q$.

Example 3.5. By X we denote the space $\ell^1 \oplus \ell^1$ with the norm $\|x\| = \|x_1 + x_2\| = \|x_1\| + \|x_2\|$. For any positive integer p let S_p be the (bounded linear) operator on ℓ^1 defined by

$$S_p(\xi_1, \xi_2, \xi_3, \dots) = (0, \xi_1, \xi_2, \xi_3, \dots, \xi_{p-1}, 0, 0, \dots),$$

and let S_∞ be the operator on ℓ^1 defined by

$$S_\infty(\xi_1, \xi_2, \xi_3, \dots) = (0, \xi_1, \tfrac{1}{2}\xi_2, \tfrac{1}{3}\xi_3, \dots).$$

We observe that S_p is nilpotent of index p, while S_∞ is a true quasinilpotent. Given a pair $p, q \in \mathbb{N} \cup \{\infty\}$, we define a (Drazin invertible) operator A on X by $A = I \oplus S_p$ and an operator $U = 0 \oplus (S_q - S_p)$. We check that A and $B = A + U$ satisfy condition (3.1): $A^{\mathrm{D}}U = (I \oplus 0)(0 \oplus (S_q - S_p)) = 0$, $A^\pi U = U = UA^\pi$ and $(A+U)A^\pi = 0 \oplus S_q$ is quasinilpotent. Then $B = I \oplus S_q$ is Drazin invertible, while $i(A) = p$ and $i(B) = q$.

We consider the perturbation of the linear equation

$$Ax = b, \quad b \in X \text{ given,} \tag{3.9}$$

(with A Drazin invertible and $x \in \mathcal{D}(A)$ to be found) in more generality than in [3, 4, 14, 18]. (In the cited references only $b \in \mathcal{K}(A)$ is considered.) We have the following result.

Theorem 3.6. *Let A, $B \in \mathcal{C}(X)$ be operators with the same domain, let A be Drazin invertible, let* (3.1) *be satisfied, and let $b, u \in X$. If $x \in X$ is a solution to $Ax = b$ and $y \in X$ is a solution to $By = b + u$, then*

$$\frac{\|Qy - Qx\|}{\|Qx\|} \le \frac{\|A^{\mathrm{D}}\|}{1 - \|UA^{\mathrm{D}}\|} \, \frac{\|UA^{\mathrm{D}}b\| + \|Qu\|}{\|A^{\mathrm{D}}b\|}, \tag{3.10}$$

where $U = B - A$ and $Q = I - A^\pi$ is the projection of X onto $\mathcal{K}(A)$ relative to the direct sum $X = \mathcal{K}(A) \oplus \mathcal{H}_0(A)$.

Proof. To prove this theorem, we proceed similarly as in the proof of [3, Theorem 3.1]. There is added generality that b, u are not assumed to lie in $\mathcal{K}(A)$ as in [3], and the operator $U = B - A$ is only linear, not necessarily defined on all of X. Recall that $UT \in \mathcal{B}(X)$ for any $T \in \mathcal{B}(X)$ with $\mathcal{R}(T) \subset \mathcal{D}(U)$. Since $QAz = AQz$ and $QBz = BQz$ for all $z \in \mathcal{D}(A)$, we first transform the equations $Ax = b$ and $By = b + u$ to $AQx = Qb$ and $BQy = Qb + Qu$, respectively. The procedure from the proof of [3, Theorem 3.1] can be now applied to yield the result when we observe that $A^{\mathrm{D}}Q = A^{\mathrm{D}}$. □

4. Error estimate using higher powers of operators

In [11] it is shown that if $A \in \mathcal{C}(X)$ is Drazin invertible, then A^k is Drazin invertible for each $k \in \mathbb{N}$, and

$$(A^k)^{\mathrm{D}} = (A^{\mathrm{D}})^k \text{ and } (A^k)^{\pi} = A^{\pi} \text{ for all } k \in \mathbb{N}. \tag{4.1}$$

Here we are interested in the converse problem. If A^m is Drazin invertible for some $m \in \mathbb{N}$, is A also Drazin invertible, and can the error bounds for the perturbation of A^{D} be calculated from the known error bounds for the perturbation of $(A^m)^{\mathrm{D}}$?

First we address the question of existence.

Lemma 4.1. *Let $A \in \mathcal{C}(X)$ be such that A^m is Drazin invertible for some $m \in \mathbb{N}$. Then A is also Drazin invertible, and* (4.1) *holds. In addition,*

$$\mathcal{D}_k(A) = \mathcal{R}((A^{\mathrm{D}})^k) + \mathcal{R}(A^{\pi}) \text{ for all } k \in \mathbb{N}. \tag{4.2}$$

Proof. We may assume that A^m is not invertible. Then 0 is an isolated spectral point of $\sigma(A^m)$. By the spectral mapping theorem for polynomials of closed operators [16, Theorem V.9.6], 0 is also isolated in $\sigma(A)$, that is, A is Drazin invertible. Equation (4.1) is satisfied in view of the above mentioned result in [11].

To prove (4.2), we note that $(A^{\mathrm{D}})^k = (A_1^k)^{-1} \oplus 0$, where $(A_1^k)^{-1}$ maps $\mathcal{K}(A)$ onto $\mathcal{K}(A) \cap \mathcal{D}_k(A)$. □

The error bounds for the perturbation of the Drazin inverse of A can be then expressed in terms of error bounds involving higher powers of A.

Theorem 4.2. *Let $m \in \mathbb{N}$ and let A, $B \in \mathcal{C}(X)$ be operators satisfying the following conditions*:

(i) $\mathcal{D}_k(B) = \mathcal{D}_k(A)$ *for* $k = m-1, m$;

(ii) A^m *is Drazin invertible*;

(iii) $\theta = \|(B^m - A^m)(A^m)^{\mathrm{D}}\| < 1$;

(iv) $A^{\pi}B^m(A^{\mathrm{D}})^m = 0 = (A^{\mathrm{D}})^m B^m A^{\pi}$;

(v) $B^m A^{\pi}$ *is quasinilpotent.*

Then the operators A and B are Drazin invertible, and

$$\frac{\|B^{\mathrm{D}} - A^{\mathrm{D}}\|}{\|A^{\mathrm{D}}\|} \le \Big(\|(B^{m-1} - A^{m-1})(A^{m-1})^{\mathrm{D}}\| + \|I - A^{\pi}\|\theta\Big)(1-\theta)^{-1}. \tag{4.3}$$

Proof. By Theorem 3.1, B^m is Drazin invertible with $(B^m)^{\mathrm{D}} = (A^m)^{\mathrm{D}}W$, where $W = (I + (B^m - A^m)(A^m)^{\mathrm{D}})^{-1}$. By the preceding lemma, both A and B are Drazin invertible and the operators $A^k(A^{\mathrm{D}})^k$, $B^k(B^{\mathrm{D}})^k$ and $B^k(A^{\mathrm{D}})^k$ are defined for $k = m-1, m$. Then

$$B^{\mathrm{D}} - A^{\mathrm{D}} = (BB^{\mathrm{D}})^{m-1}B^{\mathrm{D}} - (AA^{\mathrm{D}})^{m-1}A^{\mathrm{D}}$$

$$
\begin{aligned}
&= B^{m-1}(B^{\mathrm{D}})^m - A^{m-1}(A^{\mathrm{D}})^m \\
&= B^{m-1}(A^{\mathrm{D}})^m W - A^{m-1}(A^{\mathrm{D}})^m W + A^{m-1}(A^{\mathrm{D}})^m W - A^{m-1}(A^{\mathrm{D}})^m \\
&= (B^{m-1} - A^{m-1})(A^{m-1})^{\mathrm{D}} A^{\mathrm{D}} W + (I - A^{\pi}) A^{\mathrm{D}} W (B^m - A^m)(A^m)^{\mathrm{D}}.
\end{aligned}
$$

Taking norms, we get

$$\|B^{\mathrm{D}} - A^{\mathrm{D}}\| \le \|(B^{m-1} - A^{m-1})(A^{m-1})^{\mathrm{D}}\| \|A^{\mathrm{D}}\| (1-\theta)^{-1} + \|I - A^{\pi}\| \|A^{\mathrm{D}}\| (1-\theta)^{-1}\theta,$$

and (4.3) follows. □

Let us specialize the preceding theorem to operators in $\mathcal{B}(X)$. Then

$$\|(B^{m-1} - A^{m-1})(A^{m-1})^{\mathrm{D}}\| \le \frac{\|B^{m-1} - A^{m-1}\|}{\|A^{m-1}\|} \kappa_{\mathrm{D}}(A^{m-1}),$$

and

$$\|I - A^{\pi}\| = \|(AA^{\mathrm{D}})^{m-1}\| = \|A^{m-1}(A^{\mathrm{D}})^{m-1}\| \le \kappa_{\mathrm{D}}(A^{m-1}).$$

Inequality (4.3) then takes the following form:

$$\frac{\|B^{\mathrm{D}} - A^{\mathrm{D}}\|}{\|A^{\mathrm{D}}\|} \le \frac{\kappa_{\mathrm{D}}(A^{m-1})}{1-\theta} \left(\frac{\|B^{m-1} - A^{m-1}\|}{\|A^{m-1}\|} + \theta \right). \tag{4.4}$$

References

[1] S. L. Campbell, Continuity of the Drazin inverse, *Linear and Multilinear Algebra* **8** (1980), 265–268.
[2] S. L. Campbell and C. D. Meyer, Continuity properties of the Drazin pseudoinverse, *Linear Algebra Appl.* **10** (1975), 77–83.
[3] N. Castro González and J. J. Koliha, Perturbation of the Drazin inverse for closed linear operators, *Integral Equations Operator Theory*, in press.
[4] N. Castro González, J. J. Koliha and I. Straškraba, Perturbation of the Drazin inverse, preprint 1999.
[5] N. Castro González, J. J. Koliha and Yimin Wei, Matrices with equal eigenprojections, preprint 1999.
[6] M. P. Drazin, Pseudo-inverse in associative rings and semigroups, *Amer. Math. Monthly* **65** (1958), 506–514.
[7] J. J. Koliha, A generalized Drazin inverse, *Glasgow Math. J.* **38** (1996), 367–381.
[8] J. J. Koliha, Isolated spectral points, *Proc. Amer. Math. Soc.* **124** (1996), 3417–3424.
[9] J. J. Koliha and P. W. Poon, Spectral sets II, *Rend. Circ. Mat.* (*Palermo*) **47** (1998), 293–310.
[10] J. J. Koliha and V. Rakočević, Continuity of the Drazin inverse II, *Studia Math.* **131** (1998), 167–177.
[11] J. J. Koliha and T. D. Tran, The Drazin inverse for closed linear operators, preprint 1998.
[12] M. Mbekhta, Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux, *Glasgow Math. J.* **29** (1987), 159–175.
[13] V. Rakočević, Continuity of the Drazin inverse, *J. Operator Theory* **41** (1999), 55–68.
[14] V. Rakočević and Y. Wei, The perturbation theory for the Drazin inverse and its applications, preprint.
[15] G. Rong, The error bounds for the perturbation of the Drazin inverse, *Linear Algebra Appl.* **47** (1982). 159–168.
[16] A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, 2nd ed., Wiley, New York, 1980.
[17] Y. Wei, On the perturbation of the group inverse and oblique projection, *Appl. Math. Comput.* **98** (1999), 29–42.
[18] Y. Wei and G. Wang, The perturbation theory for the Drazin inverse and its applications, *Linear Algebra Appl.* **258** (1997), 179–186.

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