

AN INEQUALITY OF OSTROWSKI TYPE FOR TWICE DIFFERENTIABLE MAPPINGS IN TERMS OF THE L_p NORM AND APPLICATIONS

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ABSTRACT. An inequality of the Ostrowski type for twice differentiable mappings whose derivatives belong to $L_p(a, b)$, $1 < p < \infty$, and applications to special means and numerical integration are investigated.

1. INTRODUCTION

The following inequality is well known in the literature as Ostrowski's integral inequality (see for example [1, p. 468]).

Theorem 1. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $\overset{\circ}{I}$ ($\overset{\circ}{I}$ is the interior of I) and let $a, b \in \overset{\circ}{I}$ with $a < b$. If $f' : (a, b) \rightarrow \mathbb{R}$ is bounded, i.e., $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$, then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for all $x \in (a, b)$.

The constant $\frac{1}{4}$ is the best possible.

For a simple proof and some applications of Ostrowski's inequality to some special means and some numerical quadrature rules, we refer the reader to the recent paper [2] by S. S. Dragomir and S. Wang.

In [3], the same authors considered another inequality of Ostrowski type for the $\|\cdot\|_p$ -norm ($p > 1$) as follows:

Theorem 2. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $\overset{\circ}{I}$ and $a, b \in \overset{\circ}{I}$ with $a < b$. If $f' \in L_p(a, b)$ ($p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$) then we have the inequality:*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{b-a} \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right] \|f'\|_p$$

for all $x \in [a, b]$, where

$$\|f'\|_p := \left(\int_a^b |f'(t)|^p dt \right)^{\frac{1}{p}},$$

is the $L_p(a, b)$ -norm.

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They also pointed out some applications of (1.1) in Numerical Integration as well as for special means.

In 1976, G. V. Milovanović and J. E. Pečarić proved a generalization of the Ostrowski inequality for n -times differentiable mappings (see for example [1, p. 468]). The case of twice differentiable mappings [1, p. 470] is as follows:

Theorem 3. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a twice differentiable mapping such that $f'' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e., $\|f''\|_\infty := \sup_{t \in (a, b)} |f''(t)| < \infty$. Then we have the inequality:*

$$\begin{aligned} & \left| \frac{1}{2} \left[f(x) + \frac{(x-a)f(a) + (b-x)f(b)}{b-a} \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{\|f''\|_\infty}{4} (b-a)^2 \left[\frac{1}{12} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \end{aligned}$$

for all $x \in [a, b]$.

In 1998, Cerone, Dragomir and Roumeliotis [4] proved the following inequality of Ostrowski type for mappings which are twice differentiable.

Theorem 4. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a twice differentiable mapping on (a, b) and $f'' \in L_p(a, b)$ ($p > 1$). Then we have the inequality:*

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(x - \frac{a+b}{2}\right) f'(x) \right| \\ & \leq \frac{1}{2(b-a)(2q+1)^{\frac{1}{q}}} \left[(x-a)^{2q+1} + (b-x)^{2q+1} \right]^{\frac{1}{q}} \|f''\|_p \\ & \leq \frac{(b-a)^{1+\frac{1}{q}} \|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \end{aligned}$$

for all $x \in [a, b]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Dragomir and Sofo [5] proved the following inequality in the case where the second derivative belongs to the $L_\infty(\cdot, \cdot)$ norm.

Theorem 5. *Let $g : [a, b] \rightarrow \mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[a, b]$ and assume that the second derivative $g'' \in L_\infty[a, b]$. Then we have the inequality*

$$\begin{aligned} (1.2) \quad & \left| \int_a^b g(t) dt - \frac{1}{2} \left[g(x) + \frac{g(a) + g(b)}{2} \right] (b-a) + \frac{(b-a)}{2} \left(x - \frac{a+b}{2} \right) g'(x) \right| \\ & \leq \|g'\|_\infty \left(\frac{1}{3} \left| x - \frac{a+b}{2} \right|^3 + \frac{(b-a)^3}{48} \right) \end{aligned}$$

for all $x \in [a, b]$.

In this paper we point out an inequality of Ostrowski type, different to that of Cerone, Dragomir and Roumeliotis [4], for twice differentiable mappings which is in terms of the $L_p(\cdot, \cdot)$ norm of the second derivative, g'' , and apply it to special means.

2. THE MAIN THEOREM

The following theorem is now proved and later applied to special means.

Theorem 6. *Let $g : [a, b] \rightarrow \mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[a, b]$. If we assume that the second derivative $g'' \in L_p(a, b)$, $1 < p < \infty$, then we have the inequality*

$$(2.1) \quad \left| \int_a^b g(t) dt - \frac{1}{2} \left[g(x) + \frac{g(a) + g(b)}{2} \right] (b-a) + \frac{(b-a)}{2} \left(x - \frac{a+b}{2} \right) g'(x) \right|$$

$$\leq \frac{1}{2} \left(\frac{b-a}{2} \right)^{2+\frac{1}{q}} \|g''\|_p$$

$$\times \begin{cases} [B(q+1, q+1) + B_{x_1}(q+1, q+1) + \Psi_{x_2}(q+1, q+1)]^{\frac{1}{q}} \\ \text{for } x \in [a, \frac{a+b}{2}], \\ [B(q+1, q+1) + B_{x_3}(q+1, q+1) + B_{x_4}(q+1, q+1)]^{\frac{1}{q}} \\ \text{for } x \in (\frac{a+b}{2}, b], \end{cases}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, $q > 1$, and $B(\cdot, \cdot)$ is the Beta function of Euler given by

$$B(l, s) = \int_0^1 t^{l-1} (1-t)^{s-1} dt, \quad l, s > 0,$$

$$B_r(l, s) = \int_0^r t^{l-1} (1-t)^{s-1} dt$$

is the incomplete Beta function,

$$\Psi_r(l, s) = \int_0^r t^{l-1} (1+t)^{s-1} dt$$

is a real positive valued integral,

$$x_1 = \frac{2(x-a)}{b-a}, \quad x_2 = 1 - x_1,$$

$$x_3 = x_1 - 1, \quad x_4 = 2 - x_1$$

and

$$\|g''\|_p := \left(\int_a^b |g''(t)|^p dt \right)^{\frac{1}{p}}.$$

If we assume that $g'' \in L_1(a, b)$, then we have

$$(2.2) \quad \left| \int_a^b g(t) dt - \frac{1}{2} \left[g(x) + \frac{g(a) + g(b)}{2} \right] (b-a) + \frac{(b-a)}{2} \left(x - \frac{a+b}{2} \right) g'(x) \right|$$

$$\leq \frac{\|g''\|_1}{8} (b-a)^2,$$

where

$$\|g''\|_1 := \int_a^b |g''(t)| dt.$$

Proof. We begin with the proof of the following integral equality

$$(2.3) \quad f(x) = \frac{1}{b-a} \left(\int_a^b f(t) dt + \int_a^b p(x,t) f'(t) dt \right)$$

$\forall x \in [a, b]$, provided that f is absolutely continuous on $[a, b]$, and the kernel $p : [a, b]^2 \rightarrow \mathbb{R}$ is given by

$$p(x, t) := \begin{cases} t - a & \text{if } t \in [a, x], \\ t - b & \text{if } t \in (x, b]; \end{cases}$$

where $t \in [a, b]$.

A proof of (2.3) may be found in the paper by Dragomir and Wang [2].

Now, if we choose $f(x) = (x - \frac{a+b}{2})g'(x)$ and apply it in (2.3), we obtain, after a moderate amount of manipulation, details of which may be seen in a paper by Dragomir and Sofo [5], the integral equality

$$(2.4) \quad \begin{aligned} \int_a^b g(t) dt &= \frac{(b-a)}{2} \left[g(x) + \frac{g(a) + g(b)}{2} \right] \\ &\quad - \frac{(b-a)}{2} \left(x - \frac{a+b}{2} \right) g'(x) \\ &\quad + \frac{1}{2} \int_a^b p(x, t) \left(t - \frac{a+b}{2} \right) g''(t) dt, \end{aligned}$$

for all $x \in [a, b]$.

From (2.4), we have the inequality

$$(2.5) \quad \begin{aligned} &\left| \int_a^b g(t) dt - \frac{(b-a)}{2} \left[g(x) + \frac{g(a) + g(b)}{2} \right] + \frac{(b-a)}{2} \left(x - \frac{a+b}{2} \right) g'(x) \right| \\ &\leq \frac{1}{2} \left| \int_a^b p(x, t) \left(t - \frac{a+b}{2} \right) g''(t) dt \right|, \end{aligned}$$

whose left hand side is equivalent to that of (1.2).

From the right hand side of (2.5) we have, by Hölder's inequality

$$\begin{aligned} &\left| \int_a^b p(x, t) \left(t - \frac{a+b}{2} \right) g''(t) dt \right| \\ &\leq \left(\int_a^b |g''(t)|^p dt \right)^{\frac{1}{p}} \left(\int_a^b |p(x, t)|^q \left| t - \frac{a+b}{2} \right|^q dt \right)^{\frac{1}{q}} \\ &= \|g''\|_p \left(\int_a^b |p(x, t)|^q \left| t - \frac{a+b}{2} \right|^q dt \right)^{\frac{1}{q}}, \end{aligned}$$

and from (2.5) we obtain the inequality

$$(2.6) \quad \left| \int_a^b g(t) dt - \frac{(b-a)}{2} \left[g(x) + \frac{g(a)+g(b)}{2} \right] + \frac{(b-a)}{2} \left(x - \frac{a+b}{2} \right) g'(x) \right| \\ \leq \frac{1}{2} \|g''\|_p \left(\int_a^b |p(x,t)|^q \left| t - \frac{a+b}{2} \right|^q dt \right)^{\frac{1}{q}}.$$

From the right hand side of (2.6) we may define

$$(2.7) \quad I : = \int_a^b |p(x,t)|^q \left| t - \frac{a+b}{2} \right|^q dt \\ = \int_a^x (t-a)^q \left| t - \frac{a+b}{2} \right|^q dt + \int_x^b |t-b|^q \left| t - \frac{a+b}{2} \right|^q dt,$$

such that we can identify two distinct cases.

(a) For $x \in [a, \frac{a+b}{2}]$

$$I_A = \int_a^x (t-a)^q \left(\frac{a+b}{2} - t \right)^q dt + \int_x^{\frac{a+b}{2}} (b-t)^q \left(\frac{a+b}{2} - t \right)^q dt \\ + \int_{\frac{a+b}{2}}^b (b-t)^q \left(t - \frac{a+b}{2} \right)^q dt.$$

Investigating the three separate integrals, we may evaluate as follows:

$$I_1 = \int_a^x (t-a)^q \left(\frac{a+b}{2} - t \right)^q dt,$$

making the change of variable $t = a + (\frac{b-a}{2})w$, we arrive at

$$I_1 = \left(\frac{b-a}{2} \right)^{2q+1} \int_0^{x_1} w^q (1-w)^q dw \\ = \left(\frac{b-a}{2} \right)^{2q+1} B_{x_1}(q+1, q+1),$$

where $B_{x_1}(\cdot, \cdot)$ is the incomplete Beta function and $x_1 = \frac{2(x-a)}{b-a}$.

$$I_2 = \int_x^{\frac{a+b}{2}} (b-t)^q \left(\frac{a+b}{2} - t \right)^q dt,$$

making the change of variable $t = \frac{a+b}{2} - \left(\frac{b-a}{2}\right) w$, we obtain

$$\begin{aligned} I_2 &= \left(\frac{b-a}{2}\right)^{2q+1} \int_0^{x_2} w^q (1+w)^q dw \\ &= \left(\frac{b-a}{2}\right)^{2q+1} \Psi_{x_2}(q+1, q+1), \end{aligned}$$

where

$$\Psi_{x_2} := \int_0^{x_2} w^q (1+w)^q dw$$

and $x_2 = \frac{a+b-2x}{b-a} = 1 - x_1$.

$$I_3 = \int_{\frac{a+b}{2}}^b (b-t)^q \left(t - \frac{a+b}{2}\right)^q dt,$$

making the change of variable $t = \frac{a+b}{2} + \left(\frac{b-a}{2}\right) w$, we get

$$\begin{aligned} I_3 &= \left(\frac{b-a}{2}\right)^{2q+1} \int_0^1 w^q (1-w)^q dw \\ &= \left(\frac{b-a}{2}\right)^{2q+1} B(q+1, q+1), \end{aligned}$$

where $B(\cdot, \cdot)$ is the Beta function.

We may now write

$$\begin{aligned} I_A &= I_1 + I_2 + I_3 \\ &= \left(\frac{b-a}{2}\right)^{2q+1} [B_{x_1}(q+1, q+1) + \Psi_{x_2}(q+1, q+1) + B(q+1, q+1)] \end{aligned}$$

for $x \in [a, \frac{a+b}{2}]$.

(b) For $x \in (\frac{a+b}{2}, b]$

$$\begin{aligned} I_B &= \int_a^{\frac{a+b}{2}} (t-a)^q \left(\frac{a+b}{2} - t\right)^q dt + \int_{\frac{a+b}{2}}^x (t-a)^q \left(t - \frac{a+b}{2}\right)^q dt \\ &\quad + \int_x^b (b-t)^q \left(t - \frac{a+b}{2}\right)^q dt. \end{aligned}$$

In a similar fashion to the previous case, we have

$$I_4 = \int_a^{\frac{a+b}{2}} (t-a)^q \left(\frac{a+b}{2} - t\right)^q dt.$$

Letting $t = a + \left(\frac{b-a}{2}\right) w$, we obtain

$$\begin{aligned} I_4 &= \left(\frac{b-a}{2}\right)^{2q+1} \int_0^1 w^q (1-w)^q dw \\ &= \left(\frac{b-a}{2}\right)^{2q+1} B(q+1, q+1), \end{aligned}$$

where $B(\cdot, \cdot)$ is the Beta function.

$$I_5 = \int_{\frac{a+b}{2}}^x (t-a)^q \left(t - \frac{a+b}{2}\right)^q dt,$$

making the change of variable $t = \frac{a+b}{2} + \left(\frac{b-a}{2}\right)w$, we arrive at

$$\begin{aligned} I_5 &= \left(\frac{b-a}{2}\right)^{2q+1} \int_0^{x_3} w^q (1-w)^q dw \\ &= \left(\frac{b-a}{2}\right)^{2q+1} B_{x_3}(q+1, q+1), \end{aligned}$$

where $B_{x_3}(\cdot, \cdot)$ is the incomplete Beta function and $x_3 = x_1 - 1$.

$$I_6 = \int_x^b (b-t)^q \left(t - \frac{a+b}{2}\right)^q dt,$$

making the change of variable $t = b - \left(\frac{b-a}{2}\right)w$, we get

$$\begin{aligned} I_6 &= \left(\frac{b-a}{2}\right)^{2q+1} \int_0^{x_4} w^q (1-w)^q dw \\ &= \left(\frac{b-a}{2}\right)^{2q+1} B_{x_4}(q+1, q+1), \end{aligned}$$

where $B_{x_4}(\cdot, \cdot)$ is the incomplete Beta function and $x_4 = 2 - x_1$.

Now

$$\begin{aligned} I_B &= I_4 + I_5 + I_6 \\ &= \left(\frac{b-a}{2}\right)^{2q+1} [B(q+1, q+1) + B_{x_3}(q+1, q+1) + B_{x_4}(q+1, q+1)] \\ &\quad \text{for } x \in \left(\frac{a+b}{2}, b\right], \end{aligned}$$

and from (2.7)

$$\begin{aligned} I &= I_A + I_B \\ &= \left(\frac{b-a}{2}\right)^{2q+1} \begin{cases} B_{x_1}(q+1, q+1) + \Psi_{x_2}(q+1, q+1) + B(q+1, q+1) \\ \text{for } x \in [a, \frac{a+b}{2}], \\ B(q+1, q+1) + B_{x_3}(q+1, q+1) + B_{x_4}(q+1, q+1) \\ \text{for } x \in (\frac{a+b}{2}, b]. \end{cases} \end{aligned}$$

Utilizing (2.6), we obtain the result (2.1).

Using the inequality (2.5), we can also state that

$$\begin{aligned} &\left| \int_a^b g(t) dt - \frac{b-a}{2} \left[g(x) + \frac{g(a)+g(b)}{2} \right] (b-a) + \frac{(b-a)}{2} \left(x - \frac{a+b}{2} \right) g'(x) \right| \\ &\leq \frac{1}{2} \|g''\|_1 \|K(x, t)\|_\infty, \end{aligned}$$

where

$$K(x, t) := p(x, t) \left(t - \frac{a+b}{2} \right).$$

As it is easy to see that

$$\|K(x, t)\|_\infty = \frac{(b-a)^2}{4}, \quad x \in [a, b],$$

we deduce (2.2). ■

Remark 1. *The inequality (2.1) may be rewritten as follows*

$$(2.8) \quad \left| g(x) + \frac{g(a) + g(b)}{2} - \left(x - \frac{a+b}{2} \right) g'(x) - \frac{2}{b-a} \int_a^b g(t) dt \right| \\ \leq \frac{1}{2} \left(\frac{b-a}{2} \right)^{1+\frac{1}{q}} \|g''\|_p \\ \times \begin{cases} [B(q+1, q+1) + B_{x_1}(q+1, q+1) + \Psi_{x_2}(q+1, q+1)]^{\frac{1}{q}} \\ \text{for } x \in [a, \frac{a+b}{2}], \\ [B(q+1, q+1) + B_{x_3}(q+1, q+1) + B_{x_4}(q+1, q+1)]^{\frac{1}{q}} \\ \text{for } x \in (\frac{a+b}{2}, b]. \end{cases}$$

Choosing $x = a$, we obtain, from (2.8), $x_1 = 0$ and $x_2 = 1$ so that

$$(2.9) \quad \left| \frac{3g(a) + g(b)}{2} + \frac{(b-a)}{2} g'(a) - \frac{2}{b-a} \int_a^b g(t) dt \right| \\ \leq \frac{1}{2} \left(\frac{b-a}{2} \right)^{1+\frac{1}{q}} \|g''\|_p [B(q+1, q+1) + \Psi_1(q+1, q+1)]^{\frac{1}{q}}.$$

Choosing $x = b$, we obtain from (2.8) $x_3 = 1$, and $x_4 = 0$ so that

$$(2.10) \quad \left| \frac{g(a) + 3g(b)}{2} - \frac{(b-a)}{2} g'(b) - \frac{2}{b-a} \int_a^b g(t) dt \right| \\ \leq \frac{1}{4} (b-a)^{1+\frac{1}{q}} \|g''\|_p B^{\frac{1}{q}}(q+1, q+1).$$

At the midpoint $x = \frac{a+b}{2}$, we obtain the best estimator so that $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, $x_4 = 1$ and

$$\left| g\left(\frac{a+b}{2}\right) + \frac{g(a) + g(b)}{2} - \frac{2}{b-a} \int_a^b g(t) dt \right| \\ \leq \frac{1}{4} (b-a)^{1+\frac{1}{q}} \|g''\|_p B^{\frac{1}{q}}(q+1, q+1).$$

Assuming the inequalities (2.9) and (2.10), using the triangle inequality and dividing by 4, we obtain a perturbed trapezoid formula:-

$$\begin{aligned} & \left| \frac{g(a) + g(b)}{2} - \frac{(b-a)}{8} (g'(b) - g'(a)) - \frac{1}{b-a} \int_a^b g(t) dt \right| \\ & \leq \frac{1}{8} \left(\frac{b-a}{2} \right)^{1+\frac{1}{q}} \|g''\|_p \left[\left(1 + 2^{\frac{1}{q}}\right) B^{\frac{1}{q}}(q+1, q+1) + \Psi_1^{\frac{1}{q}}(q+1, q+1) \right]. \end{aligned}$$

The following particular case for Euclidean norms $p = q = 2$ is of particular importance.

Corollary 1. Let $g : [a, b] \rightarrow \mathbb{R}$ be as in Theorem 6 and $g'' \in L_2(a, b)$. Using the result (2.1), we have the inequality

$$(2.11) \quad \left| \int_a^b g(t) dt - \frac{(b-a)}{2} \left[g(x) + \frac{g(a) + g(b)}{2} \right] + \frac{(b-a)}{2} \left(x - \frac{a+b}{2} \right) g'(x) \right| \\ \leq \frac{(b-a)^{\frac{1}{2}}}{2} \|g''\|_2 \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^4 + \frac{1}{480} (b-a)^4 \right]^{\frac{1}{2}}$$

for all $x \in [a, b]$.

Proof. Applying inequality (2.1) for $p = q = 2$, we obtain, after a moderate amount of manipulation (or simply by directly integrating the expression (2.7)),

$$(2.12) \quad \left| \int_a^b g(t) dt - \frac{(b-a)}{2} \left[g(x) + \frac{g(a) + g(b)}{2} \right] + \frac{(b-a)}{2} \left(x - \frac{a+b}{2} \right) g'(x) \right| \\ \leq \frac{\|g''\|_2}{2} \left\{ \frac{(b-a)}{60} \left[30x^4 - 60x^3(a+b) + 45x^2(a+b)^2 \right. \right. \\ \left. \left. - 15x(a+b)^3 + 2a^4 + 7a^3b + 12a^2b^2 + 7ab^3 + 2b^4 \right] \right\}^{\frac{1}{2}}.$$

for $x \in [a, b]$.

Let $x := \tau + \frac{a+b}{2}$ and $A := \frac{a+b}{2}$ so that

$$\begin{aligned} 30x^4 - 120x^3A + 180x^2A^2 - 120xA^3 &= 30\tau^4 - 30A^4 \\ &= 30 \left(x - \frac{a+b}{2} \right)^4 - \frac{15}{8} (a+b)^4. \end{aligned}$$

Now, from the inner bracket of (2.12), we have

$$\begin{aligned} & 30x^4 - 120x^3A + 180x^2A^2 - 120xA^3 + 2a^4 + 7a^3b + 12a^2b^2 + 7ab^3 + 2b^4 \\ &= 30 \left(x - \frac{a+b}{2} \right)^4 + \frac{1}{8} (b-a)^4. \end{aligned}$$

and the inequality (2.11) follows. ■

3. APPLICATIONS FOR SOME SPECIAL MEANS.

Let us recall the following means:

(a) The *Arithmetic mean*:

$$A = A(a, b) := \frac{a+b}{2}, \quad a, b \geq 0.$$

(b) The *Geometric mean*:

$$G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0.$$

(c) The *Harmonic mean*:

$$H = H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad a, b > 0.$$

(d) The *Logarithmic mean*:

$$L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases}, \quad a, b > 0.$$

(e) The *Identric mean*:

$$I = I(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases}, \quad a, b > 0.$$

(f) The *p-logarithmic mean*:

$$L_p = L_p(a, b) := \begin{cases} a & \text{if } a = b \\ \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b \end{cases}, \quad a, b > 0$$

where $p \in \mathbb{R} \setminus \{-1, 0\}$. The following is well known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also well known that L_p is monotonically increasing over $p \in \mathbb{R}$ (assuming that $L_0 := I$ and $L_{-1} := L$).

The inequality (2.8) may be rewritten as:

$$(3.1) \quad \left| g(x) + \frac{g(a) + g(b)}{2} - (x - A(a, b)) g'(x) - \frac{2}{b-a} \int_a^b g(t) dt \right| \\ \leq \frac{1}{2} \left(\frac{b-a}{2} \right)^{1+\frac{1}{q}} \|g''\|_p \\ \times \begin{cases} [B(q+1, q+1) + B_{x_1}(q+1, q+1) + \Psi_{x_2}(q+1, q+1)]^{\frac{1}{q}} \\ \text{for } x \in [a, \frac{a+b}{2}], \\ [B(q+1, q+1) + B_{x_3}(q+1, q+1) + B_{x_4}(q+1, q+1)]^{\frac{1}{q}} \\ \text{for } x \in (\frac{a+b}{2}, b]. \end{cases}$$

We may now apply (3.1) to deduce some inequalities for special means given above, by the use of some particular mappings as follows.

(i). Consider $g(x) = \ln x$, $x \in [a, b] \subset (0, \infty)$. Then

$$\begin{aligned} \frac{1}{b-a} \int_a^b g(t) dt &= \ln I(a, b), \\ \frac{g(a) + g(b)}{2} &= \ln G(a, b) \end{aligned}$$

and

$$\|g''\|_p = \left(\int_a^b |g''(t)|^p dt \right)^{\frac{1}{p}} = (b-a)^{\frac{1}{p}} L_{-2p}^{-2}(a, b).$$

From (3.1),

$$\begin{aligned} & \left| \ln x + \ln G(a, b) - \left(1 - \frac{A(a, b)}{x} \right) - 2 \ln I(a, b) \right| \\ & \leq 2^{\frac{1}{p}-1} \left(\frac{b-a}{2} \right)^2 L_{-2p}^{-2}(a, b) \\ & \quad \times \begin{cases} [B(q+1, q+1) + B_{x_1}(q+1, q+1) + \Psi_{x_2}(q+1, q+1)]^{\frac{1}{q}} \\ \text{for } x \in [a, \frac{a+b}{2}], \\ [B(q+1, q+1) + B_{x_3}(q+1, q+1) + B_{x_4}(q+1, q+1)]^{\frac{1}{q}} \\ \text{for } x \in (\frac{a+b}{2}, b]. \end{cases} \end{aligned}$$

(ii). Consider $g(x) = \frac{1}{x}$, $x \in (a, b) \subset (0, \infty)$

$$\begin{aligned} \frac{1}{b-a} \int_a^b g(t) dt &= L^{-1}(a, b), \\ \frac{g(a) + g(b)}{2} &= \frac{A(a, b)}{G^2(a, b)} \end{aligned}$$

and

$$\|g''\|_p = 2(b-a)^{\frac{1}{p}} L_{-3p}^{-1}(a, b).$$

From (3.1)

$$\begin{aligned} & \left| \frac{1}{x} \left(2 - \frac{A(a, b)}{x} \right) + \frac{A(a, b)}{G^2(a, b)} - 2L^{-1}(a, b) \right| \\ & \leq 2^{\frac{1}{p}} \left(\frac{b-a}{2} \right)^2 L_{-3p}^{-1}(a, b) \\ & \quad \times \begin{cases} [B(q+1, q+1) + B_{x_1}(q+1, q+1) + \Psi_{x_2}(q+1, q+1)]^{\frac{1}{q}} \\ \text{for } x \in [a, \frac{a+b}{2}], \\ [B(q+1, q+1) + B_{x_3}(q+1, q+1) + B_{x_4}(q+1, q+1)]^{\frac{1}{q}} \\ \text{for } x \in (\frac{a+b}{2}, b]. \end{cases} \end{aligned}$$

(iii). Consider $g(x) = x^r$, $g : (0, \infty) \rightarrow \mathbb{R}$ where $r \in \mathbb{R} \setminus \{-1, 0\}$. Then, for $a < b$

$$\begin{aligned} \frac{1}{b-a} \int_a^b g(t) dt &= L_r^r(a, b), \\ \frac{g(a) + g(b)}{2} &= A(a^r, b^r) \end{aligned}$$

and

$$\|g''\|_p = |r(r-1)| (b-a)^{\frac{1}{p}} L_{p(r-2)}^{r-2}(a, b).$$

From (3.1),

$$\begin{aligned} & |x^{r-1} \{rA(a, b) + (1-r)x\} + A(a^r, b^r) - 2L_r^r(a, b)| \\ & \leq |r(r-1)| 2^{\frac{1}{p}-1} \left(\frac{b-a}{2}\right)^2 L_{p(r-2)}^{r-2}(a, b) \\ & \quad \times \begin{cases} [B(q+1, q+1) + B_{x_1}(q+1, q+1) + \Psi_{x_2}(q+1, q+1)]^{\frac{1}{q}} \\ \text{for } x \in [a, \frac{a+b}{2}], \\ [B(q+1, q+1) + B_{x_3}(q+1, q+1) + B_{x_4}(q+1, q+1)]^{\frac{1}{q}} \\ \text{for } x \in (\frac{a+b}{2}, b]. \end{cases} \end{aligned}$$

4. APPLICATIONS IN NUMERICAL INTEGRATION FOR THE $L_2(a, b)$ NORM

Let $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ be a subdivision of the interval $\xi_i \in [x_i, x_{i+1}]$, $i = 0, 1, \dots, n-1$. We have the following quadrature formula.

Theorem 7. *Let $g : [a, b] \rightarrow \mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[a, b]$ and assume that the second derivative, $g'' \in L_2(a, b)$. Then the following perturbed Riemann type quadrature formula holds.*

$$(4.1) \quad \int_a^b g(x) dx = A(g, g', \xi, I_n) + R(g, g', \xi, I_n),$$

where $A(g, g', \xi, I_n)$ is given by

$$\begin{aligned} & A(g, g', \xi, I_n) \\ & = -\frac{1}{2} \sum_{i=0}^{n-1} h_i \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) g'(\xi_i) + \frac{1}{2} \sum_{i=0}^{n-1} h_i \left[g(\xi_i) + \frac{g(x_i) + g(x_{i+1})}{2} \right] \end{aligned}$$

and the remainder $R(g, g', \xi, I_n)$ satisfies the estimation

$$|R(g, g', \xi, I_n)| \leq \frac{\|g''\|_2}{2} \left[\sum_{i=0}^{n-1} \frac{h_i}{2} \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^4 + \sum_{i=0}^{n-1} \frac{h_i^5}{480} \right]^{\frac{1}{2}}$$

for all $\xi_i \in [x_i, x_{i+1}]$.

Proof. Applying inequality (2.11) on the interval $[x_i, x_{i+1}]$, we obtain

$$\begin{aligned} & \left| \int_{x_i}^{x_{i+1}} g(t) dt - \frac{h_i}{2} \left[g(\xi_i) + \frac{g(x_i) + g(x_{i+1})}{2} \right] \right. \\ & \quad \left. + \frac{h_i}{2} \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) g'(\xi_i) \right| \\ & \leq \frac{\|g''\|_2}{2} \left\{ \frac{h_i}{60} \left[30 \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^4 + \frac{1}{8} h_i^4 \right] \right\}^{\frac{1}{2}} \end{aligned}$$

for all $i = 0, 1, \dots, n-1$.

Summing over i from 0 to $n-1$, using the triangle inequality and Cauchy-Schwartz's discrete inequality, we obtain

$$\begin{aligned} & |R(g, g', \xi, I_n)| \\ & \leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} g(t) dt - \frac{h_i}{2} \left[g(\xi_i) + \frac{g(x_i) + g(x_{i+1})}{2} \right] \right. \\ & \quad \left. + \frac{h_i}{2} \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) g'(\xi_i) \right| \\ & \leq \frac{1}{2} \sum_{i=0}^{n-1} \left\{ \frac{h_i}{60} \left[30 \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^4 + \frac{h_i^4}{8} \right] \right\}^{\frac{1}{2}} \left(\int_{x_i}^{x_{i+1}} |g''(t)|^2 dt \right)^{\frac{1}{2}} \\ & \leq \frac{1}{2} \left(\sum_{i=0}^{n-1} \left(\left\{ \frac{h_i}{60} \left[30 \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^4 + \frac{h_i^4}{8} \right] \right\}^{\frac{1}{2}} \right)^2 \right)^{\frac{1}{2}} \\ & \quad \times \left(\sum_{i=0}^{n-1} \left(\left(\int_{x_i}^{x_{i+1}} |g''(t)|^2 dt \right)^{\frac{1}{2}} \right)^2 \right)^{\frac{1}{2}} \\ & = \frac{\|g''\|_2}{2} \left(\sum_{i=0}^{n-1} \frac{h_i}{2} \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^4 + \sum_{i=0}^{n-1} \frac{h_i^5}{480} \right)^{\frac{1}{2}} \end{aligned}$$

and the theorem is proved. ■

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