AN INEQUALITY OF THE OSTROWSKI TYPE FOR DOUBLE INTEGRALS AND APPLICATIONS FOR CUBATURE FORMULAE

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Abstract. We point out a new inequality of the Ostrowski type for mappings of two independent variables, which complements, in a sense, some recent results and apply it to the approximation problem of double integrals by their Riemann sums.

1. Introduction

In the recent papers [1] and [2], the authors proved the following inequality of the Ostrowski type for double integrals.

Theorem 1. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be continuous on $[a,b]\times[c,d]$. Then we have the inequality:*

$$\begin{aligned}(1.1)\qquad & \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right.\\ & \left.-\frac{1}{d-c}\int_c^d f(x,s)\,ds-\frac{1}{b-a}\int_a^b f(t,y)\,dt+f(x,y)\right|\\ & \leq\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{4}+\left(\frac{y-\frac{c+d}{2}}{d-c}\right)^2\right](b-a)(d-c)\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty\\ \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_\infty([a,b]\times[c,d]);\\ \frac{1}{(q+1)^{\frac{2}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}\left[\left(\frac{y-c}{d-c}\right)^{q+1}+\left(\frac{d-y}{d-c}\right)^{q+1}\right]^{\frac{1}{q}}\\ \times[(b-a)(d-c)]^{\frac{1}{q}}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_p\\ \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_p([a,b]\times[c,d]),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\frac{1}{2}+\left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right]\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_1\\ \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_1([a,b]\times[c,d])\end{cases}\end{aligned}$$

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for all $(x,y)\in[a,b]\times[c,d]$, *where*

$$\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty := \sup_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|,$$

$$\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_p := \left(\int_a^b\int_c^d\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|^p dsdt\right)^{\frac{1}{p}}$$

if $p\in[1,\infty)$.

The best inequality we can get from (1.1) is the one for which $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, obtaining:

Corollary 1. *With the assumptions in Theorem 1, we have the following mid-point type inequality:*

$$\begin{aligned}(1.2)\qquad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds\right.\\ &\left.-\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\\ \le&\begin{cases}\frac{1}{16}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty(b-a)(d-c) & \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_\infty([a,b]\times[c,d]);\\[2ex] \frac{1}{4(q+1)^{\frac{2}{q}}}[(b-a)(d-c)]^{\frac{1}{q}}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_p & \\ \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_p([a,b]\times[c,d]), & \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \frac{1}{4}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_1 & \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_1([a,b]\times[c,d]).\end{cases}\end{aligned}$$

For some applications of the above results in Numerical Integration for cubature formulae see [1] and [2].

In this paper we point out some bounds for the quantity

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt-f(x,y)\right|,$$

where $(x,y)\in[a,b]\times[c,d]$ and use the results for the approximation of a double integral by its Riemann sums.

2. An Integral Identity

The following theorem holds.

Theorem 2. *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$ *be such that the partial derivatives* $\frac{\partial f(t,s)}{\partial t}$, $\frac{\partial f(t,s)}{\partial s}$, $\frac{\partial^2 f(t,s)}{\partial t\partial s}$ *exist and are continuous on* $[a,b]\times[c,d]$. *Then for all* $(x,y)\in$

$[a,b]\times[c,d]$, we have the representation

$$f(x,y) = \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(x,t)\frac{\partial f(t,s)}{\partial t}dsdt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d q(y,s)\frac{\partial f(t,s)}{\partial s}dsdt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(x,t)\,q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt, \tag{2.1}$$

where $p:[a,b]^2\to\mathbb{R}$, $q:[c,d]^2\to\mathbb{R}$ and are given by

$$p(x,t) := \begin{cases} t-a & \text{if } t\in[a,x] \\ t-b & \text{if } t\in(x,b] \end{cases}, \tag{2.2}$$

and

$$q(y,s) := \begin{cases} s-c & \text{if } s\in[c,y] \\ s-d & \text{if } s\in(y,d] \end{cases}. \tag{2.3}$$

Proof. We use the following identity, which can be easily proved by integration by parts,

$$g(u) = \frac{1}{\beta-\alpha}\int_\alpha^\beta g(z)\,dz + \frac{1}{\beta-\alpha}\int_\alpha^\beta k(u,z)\,g'(z)\,dz, \tag{2.4}$$

where $k:[\alpha,\beta]^2\to\mathbb{R}$ is given by

$$k(u,z) := \begin{cases} z-\alpha & \text{if } z\in[\alpha,u] \\ z-\beta & \text{if } z\in(u,\beta] \end{cases}$$

and g is absolutely continuous on $[\alpha,\beta]$.

Indeed, we have

$$\int_\alpha^u (z-\alpha)\,g'(z)\,dz = (u-\alpha)\,g(u) - \int_\alpha^u g(z)\,dz$$

and

$$\int_u^\beta (z-\beta)\,g'(z)\,dz = (\beta-u)\,g(u) - \int_u^\beta g(z)\,dz$$

which produces, by summation, the desired identity (2.4).

Now, write the identity (2.4) for the partial map $f(\cdot,y)$, $y\in[c,d]$ to obtain

$$f(x,y) = \frac{1}{b-a}\int_a^b f(t,y)\,dt + \frac{1}{b-a}\int_a^b p(x,t)\frac{\partial f(t,y)}{\partial t}dt \tag{2.5}$$

for all $(x,y)\in[a,b]\times[c,d]$.

Also, if we write (2.4) for the map $f(t,\cdot)$, we get

$$f(t,y) = \frac{1}{d-c}\int_c^d f(t,s)\,ds + \frac{1}{d-c}\int_c^d q(y,s)\frac{\partial f(t,s)}{\partial s}ds, \tag{2.6}$$

for all $(t,y)\in[a,b]\times[c,d]$.

The same formula (2.4) applied for the partial derivative $\frac{\partial f(\cdot,y)}{\partial t}$ will produce

$$\frac{\partial f(t,y)}{\partial t}=\frac{1}{d-c}\int_c^d\frac{\partial f(t,s)}{\partial t}ds+\frac{1}{d-c}\int_c^d q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}ds \tag{2.7}$$

for all $(t,y)\in[a,b]\times[c,d]$.

Substituting (2.6) and (2.7) in (2.5), and using Fubini's theorem, we have

$$\begin{aligned}
f(x,y) &= \frac{1}{b-a}\int_a^b\left[\frac{1}{d-c}\int_c^d f(t,s)\,ds+\frac{1}{d-c}\int_c^d q(y,s)\frac{\partial f(t,s)}{\partial s}ds\right]dt\\
&\quad+\frac{1}{b-a}\int_a^b p(x,t)\left[\frac{1}{d-c}\int_c^d\frac{\partial f(t,s)}{\partial t}ds\right.\\
&\quad\left.+\frac{1}{d-c}\int_c^d q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}ds\right]dt\\
&= \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d f(t,s)\,dsdt+\int_a^b\int_c^d q(y,s)\frac{\partial f(t,s)}{\partial s}dsdt\right.\\
&\quad\left.+\int_a^b\int_c^d p(x,t)\frac{\partial f(t,s)}{\partial t}dsdt+\int_a^b\int_c^d p(x,t)\,q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right],
\end{aligned}$$

and the identity (2.1) is completely proved. ∎

A particular case which is of interest is embodied in the following corollary.

Corollary 2. *Let f be as in Theorem 2, then we have the identity*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
&= \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d f(t,s)\,dsdt+\int_a^b\int_c^d p_0(t)\frac{\partial f(t,s)}{\partial t}dsdt\right.\\
&\quad\left.+\int_a^b\int_c^d q_0(s)\frac{\partial f(t,s)}{\partial s}dsdt+\int_a^b\int_c^d p_0(t)\,q_0(s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right],
\end{aligned} \tag{2.8}$$

where $p_0:[a,b]\to\mathbb{R}$, $q_0:[c,d]\to\mathbb{R}$ are given by

$$p_0(t):=\begin{cases} t-a & \text{if } t\in\left[a,\frac{a+b}{2}\right] \\ t-b & \text{if } t\in\left(\frac{a+b}{2},b\right]\end{cases},$$

and

$$q_0(s):=\begin{cases} s-c & \text{if } s\in\left[c,\frac{c+d}{2}\right] \\ s-d & \text{if } s\in\left(\frac{c+d}{2},d\right]\end{cases}.$$

The following corollary which provides a trapezoid type identity is also of interest.

Corollary 3. *Let f be as in Theorem 2. Then we have the identity*

$$\begin{aligned}
(2.9) \qquad & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
= \; & \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d f(t,s)\,dsdt+\int_a^b\int_c^d\left(t-\frac{a+b}{2}\right)\frac{\partial f(t,s)}{\partial t}dsdt\right. \\
& +\int_a^b\int_c^d\left(s-\frac{c+d}{2}\right)\frac{\partial f(t,s)}{\partial s}dsdt \\
& \left.+\int_a^b\int_c^d\left(t-\frac{a+b}{2}\right)\left(s-\frac{c+d}{2}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right].
\end{aligned}$$

Proof. Letting $(x,y)=(a,c)\,,(a,d)\,,(b,c)$ and (b,d) in (2.1) , we obtain successively

$$\begin{aligned}
f(a,c) = \; & \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d f(t,s)\,dsdt+\int_a^b\int_c^d(t-b)\frac{\partial f(t,s)}{\partial t}dsdt\right. \\
& \left.+\int_a^b\int_c^d(s-d)\frac{\partial f(t,s)}{\partial s}dsdt+\int_a^b\int_c^d(t-b)(s-d)\frac{\partial^2 f(t,s)}{\partial s\partial t}dsdt\right],
\end{aligned}$$

$$\begin{aligned}
f(a,d) = \; & \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d f(t,s)\,dsdt+\int_a^b\int_c^d(t-b)\frac{\partial f(t,s)}{\partial t}dsdt\right. \\
& \left.+\int_a^b\int_c^d(s-c)\frac{\partial f(t,s)}{\partial s}dsdt+\int_a^b\int_c^d(t-b)(s-c)\frac{\partial^2 f(t,s)}{\partial s\partial t}dsdt\right],
\end{aligned}$$

$$\begin{aligned}
f(b,c) = \; & \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d f(t,s)\,dsdt+\int_a^b\int_c^d(t-a)\frac{\partial f(t,s)}{\partial t}dsdt\right. \\
& \left.+\int_a^b\int_c^d(s-d)\frac{\partial f(t,s)}{\partial s}dsdt+\int_a^b\int_c^d(t-a)(s-d)\frac{\partial^2 f(t,s)}{\partial s\partial t}dsdt\right],
\end{aligned}$$

and

$$\begin{aligned}
f(b,d) = \; & \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d f(t,s)\,dsdt+\int_a^b\int_c^d(t-a)\frac{\partial f(t,s)}{\partial t}dsdt\right. \\
& \left.+\int_a^b\int_c^d(s-c)\frac{\partial f(t,s)}{\partial s}dsdt+\int_a^b\int_c^d(t-a)(s-c)\frac{\partial^2 f(t,s)}{\partial s\partial t}dsdt\right].
\end{aligned}$$

After summing over the above equalities, dividing by 4 and some simple computation, we arrive at the desired identity (2.9). ∎

3. Some Inequalities

We can state the following inequality of the Ostrowski type which holds for mappings of two independent variables.

Theorem 3. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a mapping as in Theorem 2. Then we have the inequality:*

$$\left|f(x,y)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,ds\,dt\right| \leq M_1(x)+M_2(y)+M_3(x,y), \tag{3.1}$$

where

$$M_1(x)=\begin{cases}\frac{\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]}{b-a}\left\|\frac{\partial f}{\partial t}\right\|_\infty, & \text{if } \frac{\partial f(t,s)}{\partial t}\in L_\infty([a,b]\times[c,d]);\\[2ex] \frac{\left[\frac{(b-x)^{q_1+1}+(x-a)^{q_1+1}}{q_1+1}\right]^{\frac{1}{q_1}}}{(b-a)[(d-c)]^{\frac{1}{p_1}}}\left\|\frac{\partial f}{\partial t}\right\|_{p_1}, & \text{if } \frac{\partial f(t,s)}{\partial t}\in L_{p_1}([a,b]\times[c,d]),\\ & p_1>1,\ \frac{1}{p_1}+\frac{1}{q_1}=1;\\ \frac{\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]}{(b-a)(d-c)}\left\|\frac{\partial f}{\partial t}\right\|_1, & \text{if } \frac{\partial f(t,s)}{\partial t}\in L_1([a,b]\times[c,d]).\end{cases}$$

$$M_2(y)=\begin{cases}\frac{\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]}{d-c}\left\|\frac{\partial f}{\partial s}\right\|_\infty, & \text{if } \frac{\partial f(t,s)}{\partial s}\in L_\infty([a,b]\times[c,d]);\\[2ex] \frac{\left[\frac{(d-y)^{q_2+1}+(y-c)^{q_2+1}}{q_2+1}\right]^{\frac{1}{q_2}}}{[(b-a)]^{\frac{1}{p_2}}(d-c)}\left\|\frac{\partial f}{\partial s}\right\|_{p_2}, & \text{if } \frac{\partial f(t,s)}{\partial s}\in L_{p_2}([a,b]\times[c,d]);\\ & p_2>1,\ \frac{1}{p_2}+\frac{1}{q_2}=1;\\ \frac{\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]}{(b-a)(d-c)}\left\|\frac{\partial f}{\partial s}\right\|_1, & \text{if } \frac{\partial f(t,s)}{\partial s}\in L_1([a,b]\times[c,d]);\end{cases}$$

and

$$M_3(x,y)=\begin{cases}\frac{\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]}{(b-a)(d-c)}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty,\\ \text{if } \frac{\partial^2 f(t,s)}{\partial s\partial t}\in L_\infty([a,b]\times[c,d]);\\[2ex] \frac{\left[\frac{(b-x)^{q_3+1}+(x-a)^{q_3+1}}{q_3+1}\right]^{\frac{1}{q_3}}\left[\frac{(d-y)^{q_3+1}+(y-c)^{q_3+1}}{q_3+1}\right]^{\frac{1}{q_3}}}{(b-a)(d-c)}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{p_3},\\ \text{if } \frac{\partial^2 f(t,s)}{\partial s\partial t}\in L_{p_3}([a,b]\times[c,d]),\ p_3>1,\ \frac{1}{p_3}+\frac{1}{q_3}=1;\\[2ex] \frac{\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]}{(b-a)(d-c)}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_1,\\ \text{if } \frac{\partial^2 f(t,s)}{\partial s\partial t}\in L_1([a,b]\times[c,d]);\end{cases}$$

for all $(x,y)\in[a,b]\times[c,d]$, where $\|\cdot\|_p$ $(1\leq p\leq\infty)$ are the usual $p-$norms on $[a,b]\times[c,d]$.

Proof. Using the identity (2.1) , we can state that

$$
\begin{aligned}
(3.2)\quad & \left| f(x,y) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\, dsdt \right| \\
\leq\; & \left| \frac{1}{(b-a)(d-c)} \left[\int_a^b \int_c^d p(x,t) \frac{\partial f(t,s)}{\partial t} dsdt \right.\right. \\
& \left.\left. + \int_a^b \int_c^d q(y,s) \frac{\partial f(t,s)}{\partial s} dsdt + \int_a^b \int_c^d p(x,t)\, q(y,s) \frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt \right] \right| \\
\leq\; & \frac{1}{(b-a)(d-c)} \left[\int_a^b \int_c^d |p(x,t)| \left| \frac{\partial f(t,s)}{\partial t} \right| dsdt \right. \\
& \left. + \int_a^b \int_c^d |q(y,s)| \left| \frac{\partial f(t,s)}{\partial s} \right| dsdt + \int_a^b \int_c^d |p(x,t)|\, |q(y,s)| \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right| dsdt \right].
\end{aligned}
$$

We have that

$$
\begin{aligned}
(3.3)\quad & \int_a^b \int_c^d |p(x,t)| \left| \frac{\partial f(t,s)}{\partial t} \right| dsdt \\
\leq & \begin{cases}
\left\| \frac{\partial f}{\partial t} \right\|_\infty \int_a^b \int_c^d |p(x,t)|\, dsdt, & \text{if } \frac{\partial f(t,s)}{\partial t} \in L_\infty([a,b] \times [c,d]); \\[2mm]
\left\| \frac{\partial f}{\partial t} \right\|_{p_1} \left(\int_a^b \int_c^d |p(x,t)|^{q_1} dsdt \right)^{\frac{1}{q_1}}, & \text{if } \frac{\partial f(t,s)}{\partial t} \in L_{p_1}([a,b] \times [c,d]), \\
 & p_1 > 1,\ \frac{1}{p_1} + \frac{1}{q_1} = 1; \\
\left\| \frac{\partial f}{\partial t} \right\|_1 \sup_{t \in [a,b]} |p(x,t)|, & \text{if } \frac{\partial f(t,s)}{\partial t} \in L_1([a,b] \times [c,d]).
\end{cases}
\end{aligned}
$$

and as

$$
\begin{aligned}
\int_a^b \int_c^d |p(x,t)|\, dsdt &= \int_c^d \left(\int_a^b |p(x,t)|\, dt \right) ds \\
&= (d-c) \left[\int_a^x |p(x,t)|\, dt + \int_x^b |p(x,t)|\, dt \right] \\
&= (d-c) \left[\int_a^x (t-a)\, dt + \int_x^b (b-t)\, dt \right] \\
&= (d-c) \left[\frac{(x-a)^2 + (b-x)^2}{2} \right] \\
&= (d-c) \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right],
\end{aligned}
$$

$$\begin{aligned}\left[\int_a^b\int_c^d |p(x,t)|^{q_1}\,dsdt\right]^{\frac{1}{q_1}} &= \left[\int_c^d\left(\int_a^b |p(x,t)|^{q_1}\,dt\right)ds\right]^{\frac{1}{q_1}}\\ &= (d-c)^{\frac{1}{q_1}}\left[\int_a^x |p(x,t)|^{q_1}\,dt+\int_x^b |p(x,t)|^{q_1}\,dt\right]^{\frac{1}{q_1}}\\ &= (d-c)^{\frac{1}{q_1}}\left[\int_a^x (t-a)^{q_1}\,dt+\int_x^b (b-t)^{q_1}\,dt\right]^{\frac{1}{q_1}}\\ &= (d-c)^{\frac{1}{q_1}}\left[\frac{(b-x)^{q_1+1}+(x-a)^{q_1+1}}{q_1+1}\right]^{\frac{1}{q_1}}\end{aligned}$$

and

$$\sup_{t\in[a,b]}|p(x,t)| = \max\{x-a,b-x\} = \frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|,$$

then, by (3.3), we obtain

$$\int_a^b\int_c^d |p(x,t)|\left|\frac{\partial f(t,s)}{\partial t}\right|dsdt \tag{3.4}$$

$$\leq\begin{cases}(d-c)\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left\|\frac{\partial f}{\partial t}\right\|_\infty, & \text{if } \frac{\partial f(t,s)}{\partial t}\in L_\infty([a,b]\times[c,d]);\\ (d-c)^{\frac{1}{q_1}}\left[\frac{(b-x)^{q_1+1}+(x-a)^{q_1+1}}{q_1+1}\right]^{\frac{1}{q_1}}\left\|\frac{\partial f}{\partial t}\right\|_{p_1}, & \text{if } \frac{\partial f(t,s)}{\partial t}\in L_{p_1}([a,b]\times[c,d]),\\ & p_1>1,\ \frac{1}{p_1}+\frac{1}{q_1}=1;\\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left\|\frac{\partial f}{\partial t}\right\|_1. & \text{if } \frac{\partial f(t,s)}{\partial t}\in L_1([a,b]\times[c,d]).\end{cases}$$

In a similar fashion, we can state that the following inequality holds

$$\int_a^b\int_c^d |q(y,s)|\left|\frac{\partial f(t,s)}{\partial s}\right|dsdt \tag{3.5}$$

$$\leq\begin{cases}(b-a)\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]\left\|\frac{\partial f}{\partial s}\right\|_\infty, & \text{if } \frac{\partial f(t,s)}{\partial s}\in L_\infty([a,b]\times[c,d]);\\ (b-a)^{\frac{1}{q_2}}\left[\frac{(d-y)^{q_2+1}+(y-c)^{q_2+1}}{q_2+1}\right]^{\frac{1}{q_2}}\left\|\frac{\partial f}{\partial s}\right\|_{p_2}, & \text{if } \frac{\partial f(t,s)}{\partial s}\in L_{p_2}([a,b]\times[c,d]),\\ & p_2>1,\ \frac{1}{p_2}+\frac{1}{q_2}=1;\\ \left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]\left\|\frac{\partial f}{\partial s}\right\|_1. & \text{if } \frac{\partial f(t,s)}{\partial s}\in L_1([a,b]\times[c,d]).\end{cases}$$

In addition, we have

$$\int_a^b\int_c^d |p(x,t)|\,|q(y,s)|\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|dsdt \tag{3.6}$$

$$\leq \begin{cases} \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty \int_a^b |p(x,t)|\, dt \int_c^d |q(y,s)|\, ds, \\ \text{if} \quad \frac{\partial^2 f(t,s)}{\partial s \partial t} \in L_\infty([a,b]\times[c,d]); \\ \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_{p_3} \left(\int_a^b |p(x,t)|^{q_3}\, dt \right)^{\frac{1}{q_3}} \left(\int_c^d |q(y,s)|^{q_3}\, ds \right)^{\frac{1}{q_3}}, \\ \text{if} \quad \frac{\partial^2 f(t,s)}{\partial s \partial t} \in L_{p_3}([a,b]\times[c,d]),\ p_3 > 1,\ \frac{1}{p_3} + \frac{1}{q_3} = 1; \\ \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_1 \sup_{t\in[a,b]} |p(x,t)| \sup_{s\in[c,d]} |q(y,s)| \\ \text{if} \quad \frac{\partial^2 f(t,s)}{\partial s \partial t} \in L_1([a,b]\times[c,d]). \end{cases}$$

$$= \begin{cases} \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2 \right] \left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2}\right)^2 \right] \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty, \\ \text{if} \quad \frac{\partial^2 f(t,s)}{\partial s \partial t} \in L_\infty([a,b]\times[c,d]); \\ \left[\frac{(b-x)^{q_3+1} + (x-a)^{q_3+1}}{q_3+1} \right]^{\frac{1}{q_3}} \left[\frac{(d-y)^{q_3+1} + (y-c)^{q_3+1}}{q_3+1} \right]^{\frac{1}{q_3}} \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_{p_3}, \\ \text{if} \quad \frac{\partial^2 f(t,s)}{\partial s \partial t} \in L_{p_3}([a,b]\times[c,d]);\ p_3 > 1,\ \frac{1}{p_3} + \frac{1}{q_3} = 1; \\ \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \left[\frac{1}{2}(d-c) + \left| y - \frac{c+d}{2} \right| \right] \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_1, \\ \text{if} \quad \frac{\partial^2 f(t,s)}{\partial s \partial t} \in L_1([a,b]\times[c,d]). \end{cases}$$

and the theorem is proved. ∎

The following corollary holds by taking $x = \frac{a+b}{2}$, $y = \frac{c+d}{2}$.

Corollary 4. *With the assumptions in Theorem 2, we have the inequality*

$$\left| f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\, ds dt \right| \leq \tilde{M}_1 + \tilde{M}_2 + \tilde{M}_3, \tag{3.7}$$

where

$$\tilde{M}_1 := \begin{cases} \frac{1}{4}(b-a) \left\| \frac{\partial f}{\partial t} \right\|_\infty, & \text{if} \quad \frac{\partial f}{\partial t} \in L_\infty([a,b]\times[c,d]) \\ \frac{1}{2} \left[\frac{(b-a)^{\frac{1}{q_1}}}{(q_1+1)^{\frac{1}{q_1}} (d-c)^{\frac{1}{p_1}}} \right] \left\| \frac{\partial f}{\partial t} \right\|_{p_1}, & \text{if} \quad \frac{\partial f}{\partial t} \in L_{p_1}([a,b]\times[c,d]) \\ & \quad p_1 > 1,\ \frac{1}{p_1} + \frac{1}{q_1} = 1; \\ \frac{1}{2(d-c)} \left\| \frac{\partial f}{\partial t} \right\|_1, & \text{if} \quad \frac{\partial f}{\partial t} \in L_1([a,b]\times[c,d]) \end{cases}$$

$$\tilde{M}_2 := \begin{cases} \frac{1}{4}(d-c)\left\|\frac{\partial f}{\partial s}\right\|_\infty, & \text{if } \frac{\partial f}{\partial t} \in L_\infty([a,b]\times[c,d]) \\ \frac{1}{2}\left[\frac{(d-c)^{\frac{1}{q_2}}}{(q_2+1)^{\frac{1}{q_2}}(b-a)^{\frac{1}{p_2}}}\right]\left\|\frac{\partial f}{\partial s}\right\|_{p_2} & \text{if } \frac{\partial f}{\partial t} \in L_{p_2}([a,b]\times[c,d]) \\ & p_2>1,\ \frac{1}{p_2}+\frac{1}{q_2}=1; \\ \frac{1}{2(b-a)}\left\|\frac{\partial f}{\partial s}\right\|_1, & \text{if } \frac{\partial f}{\partial t} \in L_1([a,b]\times[c,d]) \end{cases}$$

and

$$\tilde{M}_3 := \begin{cases} \frac{1}{16}(b-a)(d-c)\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty, & \text{if } \frac{\partial^2 f}{\partial t\partial s} \in L_\infty([a,b]\times[c,d]); \\ \frac{1}{4}\cdot\frac{(b-a)^{\frac{1}{q_3}}(d-c)^{\frac{1}{q_3}}}{(q_3+1)^{\frac{2}{q_3}}}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{p_3}, & \text{if } \frac{\partial^2 f}{\partial t\partial s} \in L_{p_3}([a,b]\times[c,d]); \\ & p_3>1,\ \frac{1}{p_3}+\frac{1}{q_3}=1, \\ \frac{1}{4}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_1, & \text{if } \frac{\partial^2 f}{\partial t\partial s} \in L_1([a,b]\times[c,d]). \end{cases}$$

Using the inequality (2.9) in Corollary 3 and a similar argument to the one used in Theorem 3, we can point out the following trapezoid type inequality:

Corollary 5. *With the assumption in Theorem 2, we have the inequality*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right| \leq \tilde{M}_1+\tilde{M}_2+\tilde{M}_3, \tag{3.8}$$

where the $\tilde{M}_i \ \ (i=1,2,3)$ *are as given above.*

4. Applications for Cubature Formulae

Let us consider the arbitrary divisions $I_n := a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ and $J_m : c = y_0 < y_1 < ... < y_{m-1} < y_m = d$, where $\xi_i \in [x_i, x_{i+1}]$ $(i=0,...,n-1)$, $\eta_j \in [y_j, y_{j+1}]$ $(j=0,...,m-1)$ are intermediate points. Consider the Riemann sum:

$$R(f, I_n, J_m, \xi, \eta) = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i\, l_j\, f(\xi_i, \eta_j), \tag{4.1}$$

where $h_i := x_{i+1}-x_i,\ l_j := y_{j+1}-y_j,\ i=0,...,n-1,\ j=0,...,m-1.$

Using Theorem 3, we can state twenty-seven different inequalities in bounding the quantity

$$\left|f(x,y) - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right|, \quad (x,y)\in[a,b]\times[c,d]. \tag{4.2}$$

Let us consider only one case, namely, when all the partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial s}$, $\frac{\partial^2 f}{\partial t \partial s}$ are bounded. That is,

$$\left| f(x,y) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\,dsdt \right| \tag{4.3}$$
$$\leq \frac{1}{b-a}\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \left\| \frac{\partial f}{\partial t} \right\|_\infty$$
$$+ \frac{1}{d-c}\left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2}\right)^2\right] \left\| \frac{\partial f}{\partial s} \right\|_\infty$$
$$+ \frac{1}{(b-a)(d-c)}\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]$$
$$\times \left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2}\right)^2\right] \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty$$

for all $[x,y] \in [a,b] \times [c,d]$.

Using this inequality, we can state the following theorem.

Theorem 4. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be as in Theorem 2. Then we have*

$$\int_a^b \int_c^d f(t,s)\,dsdt = R(f, I_n, J_m, \xi, \eta) + W(f, I_n, J_m, \xi, \eta), \tag{4.4}$$

where $R(f, I_n, J_m, \xi, \eta)$ is the Riemann sum defined by (4.1) and the remainder through the approximation $W(f, I_n, J_m, \xi, \eta)$ satisfies the bound

$$|W(f, I_n, J_m, \xi, \eta)| \tag{4.5}$$
$$\leq (d-c)\left\| \frac{\partial f}{\partial t} \right\|_\infty \sum_{i=0}^{n-1} \left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]$$
$$+ (b-a)\left\| \frac{\partial f}{\partial s} \right\|_\infty \sum_{j=0}^{m-1} \left[\frac{1}{4}l_j^2 + \left(\eta_j - \frac{y_j + y_{j+1}}{2}\right)^2\right]$$
$$+ \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \sum_{i=0}^{n-1} \left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]$$
$$\times \sum_{j=0}^{m-1} \left[\frac{1}{4}l_j^2 + \left(\eta_j - \frac{y_j + y_{j+1}}{2}\right)^2\right]$$
$$\leq \frac{1}{2}(d-c)\left\| \frac{\partial f}{\partial t} \right\|_\infty \sum_{i=0}^{n-1} h_i^2 + \frac{1}{2}(b-a)\left\| \frac{\partial f}{\partial s} \right\|_\infty \sum_{j=0}^{m-1} l_j^2$$
$$+ \frac{1}{4}\left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \sum_{i=0}^{n-1} h_i^2 \sum_{j=0}^{m-1} l_j^2$$
$$\leq \frac{1}{2}(d-c)(b-a)\left[\nu(h)\left\| \frac{\partial f}{\partial t} \right\|_\infty + \nu(l)\left\| \frac{\partial f}{\partial s} \right\|_\infty + \frac{1}{2}\left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \nu(h)\,\nu(l)\right]$$

for all ξ, η intermediate points, where $\nu(h) := \max\{h_i, i = 0, ..., n-1\}$ and $\nu(l) :=$ $\max\{l_j, j = 0, ..., m-1\}$.

Proof. Apply (4.3) in the intervals $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ to obtain

$$\begin{aligned}
& \left| \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f(t,s)\, ds dt - h_i l_j f(\xi_i, \eta_j) \right| \\
\leq & \left[\frac{1}{4} h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right] l_j \left\| \frac{\partial f}{\partial t} \right\|_\infty \\
& + \left[\frac{1}{4} l_j^2 + \left(\eta_j - \frac{y_j + y_{j+1}}{2} \right)^2 \right] h_i \left\| \frac{\partial f}{\partial s} \right\|_\infty \\
& + \left[\frac{1}{4} h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \left[\frac{1}{4} l_j^2 + \left(\eta_j - \frac{y_j + y_{j+1}}{2} \right)^2 \right] \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty
\end{aligned}$$

for all $i = 0, ..., n-1,\ j = 0, ..., m-1$.

Summing over i from 0 to $n-1$ and over j from 0 to $m-1$, we get the desired estimation (4.5). We omit the details. ∎

Consider the mid-point formula:

$$M(f, I_n, J_m) := \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i\, l_j\, f\left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2} \right). \tag{4.6}$$

The following corollary contains the best quadrature formula we can obtain from (4.5).

Corollary 6. *Let f be as in Theorem 2, then we have:*

$$\int_a^b \int_c^d f(t,s)\, ds dt = M(f, I_n, J_m) + L(f, I_n, J_m), \tag{4.7}$$

where $M(f, I_n, J_m)$ is the midpoint formula given by (4.6) and the remainder $L(f, I_n, J_m)$ satisfies the estimate

$$\begin{aligned}
& |L(f, I_n, J_m)| \\
\leq & \frac{1}{4}(d-c) \left\| \frac{\partial f}{\partial t} \right\|_\infty \sum_{i=0}^{n-1} h_i^2 + \frac{1}{4}(b-a) \left\| \frac{\partial f}{\partial s} \right\|_\infty \sum_{j=0}^{m-1} l_j^2 \\
& + \frac{1}{16} \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \sum_{i=0}^{n-1} h_i^2 \sum_{j=0}^{m-1} l_j^2 \\
: & = \mathcal{M}_1(f, I_n, J_m) \\
\leq & \frac{1}{4}(d-c)(b-a) \left[\nu(h) \left\| \frac{\partial f}{\partial t} \right\|_\infty + \nu(l) \left\| \frac{\partial f}{\partial s} \right\|_\infty + \frac{1}{4} \nu(h) \nu(l) \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \right] \\
: & = \mathcal{M}_2(f, I_n, J_m).
\end{aligned} \tag{4.8}$$

We can also consider the trapezoid formula

$$\begin{aligned}
& T(f, I_n, J_m) \\
: & = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i\, l_j \cdot \frac{f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})}{4}.
\end{aligned} \tag{4.9}$$

Using Corollary 5 and a similar argument to the one used in the proof of Theorem 4, we can state the following corollary.

Corollary 7. *Let f be as in Theorem 2, then we have*

$$\int_a^b \int_c^d f(t,s)\,dsdt = T(f, I_n, J_m) + P(f, I_n, J_m), \tag{4.10}$$

where $T(f, I_n, J_m)$ is the trapezoid formula obtained by (4.9) *and the remainder $P(f, I_n, J_m)$ satisfies the estimate*

$$|P(f, I_n, J_m)| \leq \mathcal{M}_1(f, I_n, J_m) \leq \mathcal{M}_2(f, I_n, J_m), \tag{4.11}$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are as defined above.

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