A VARIATIONAL CHARACTERIZATION OF REFLEXIVITY AND STRICT CONVEXITY

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Abstract. In this paper we give a variational characterization of reflexivity and strict convexity which is related to James and Krein theorems in Geometry of Banach spaces.

1. Introduction

Let $(X, \|\cdot\|)$ be a real normed space and consider the norm derivatives

$$(x, y)_{i(s)} = \lim_{t \to -(+)0} \frac{(\|y + tx\|^2 - \|y\|^2)}{2t}.$$

Note that these mappings are well defined on $X \times X$ and the following properties are valid (see also [1]-[5]):

(i) $(x, y)_i = -(-x, y)_s$ if x, y are in X;
(ii) $(x, x)_p = \|x\|^2$ for all x in X;
(iii) $(\alpha x, \beta y)_p = \alpha\beta\,(x, y)_p$ for all x, y in X and $\alpha\beta \geq 0$;
(iv) $(\alpha x + y, x)_p = \alpha\|x\|^2 + (y, x)_p$ for all x, y in X and α a real number;
(v) $(x + y, z)_p \leq \|x\| \cdot \|z\| + (y, z)_p$ for all x, y, z in X;
(vi) The element x in X is Birkhoff orthogonal over y in X (we denote this by $x \perp y$), i.e., $\|x + ty\| \geq \|x\|$ for all t a real number iff $(y, x)_i \leq 0 \leq (y, x)_s$;
(vii) The space X is smooth iff $(y, x)_i = (y, x)_s$ for all x, y in X iff $(\cdot, \cdot)_p$ is linear in the first variable;

where $p = s$ or $p = i$.

The following characterization of reflexivity is well known (see [6]):

Theorem 1. ***(James).*** *The Banach space X is reflexive iff for any continuous linear functional $f : X \to \mathbb{R}$ there exists an element u_f in X such that $f(u_f) = \|f\| \cdot \|u_f\|$, i.e., u_f is a maximal element for f.*

The following characterization of strict convexity in terms of maximal elements is well known and is due to M. G. Krein ([7, p. 27]):

Theorem 2. ***(Krein).*** *The real Banach space X is strictly convex iff any nonzero continuous linear functional defined on it has at most one maximal element having a norm equal to one.*

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2. The Results

We give here a variational characterization of reflexivity and strict convexity as follows.

Theorem 3. *Let $(X, \|\cdot\|)$ be a real Banach space. The following statements are equivalent:*

(i) *X is reflexive [strictly convex (reflexive and strictly convex)];*

(ii) *For any nonzero continuous linear functional $f : X \to \mathbb{R}$ there exists at least one [at most one (a unique)] vector $u_f \in X$, $\|u_f\| = 1$ which minimizes the quadratic functional $F_f : X \to \mathbb{R}$, $F_f(x) = \|x\|^2 - \frac{2f(x)}{\|f\|}$.*

Proof. "(i) ⇒ (ii)" a). Assume that X is reflexive and let $f \in X^*\backslash\{0\}$. Then by James' theorem there exists a vector $u_f \in X, \|u_f\| = 1$ such that $f(u_f) = \|f\|$. However,

$$\|u_f\| = 1 = \frac{f(u_f)}{\|f\|} = \frac{f(u_f + \lambda u)}{\|f\|} \le \|u_f + \lambda u\|$$

for all $\lambda \in \mathbb{R}$ and $u \in Ker(f)$, which gives us that $u_f \perp Ker(f)$.
Let $x \in X$ be arbitrary but fixed and define $y := f(x)\, u_f - f(u_f)\, x$. As $f(y) = 0$, we get that $y \in Ker(f)$ and then $u_f \perp y$ in Birkhoff's sense. By the property (vi) we get that

$$(y, x)_i \le 0 \le (y, x)_s \tag{2.1}$$

which is equivalent with

$$(f(x)\, u_f - f(u_f)\, x, u_f)_i \le 0 \le (f(x)\, u_f - f(u_f)\, x, u_f)_s \text{ for all } x \in X.$$

Using the properties of semi-inner products we get

$$(f(x)\, u_f - f(u_f)\, x, u_f)_i = f(x) - \|f\|\,(x, u_f)_s$$

and

$$(f(x)\, u_f - f(u_f)\, x, u_f)_s = f(x) - \|f\|\,(x, u_f)_i$$

for all $x \in X$, and then by (2.1) we get that

$$\|f\|\,(x, u_f)_i \le f(x) \le \|f\|\,(x, u_f)_s \text{ for all } x \in X. \tag{2.2}$$

We shall prove now that u_f minimizes the quadratic functional F_f.
Let $u \in X$. Then, as $f(u_f) = \|f\|$ and $\|u_f\| = 1$, we get that

$$\begin{aligned} F_f(u) - F_f(u_f) &= \|u\|^2 - \frac{2f(u)}{\|f\|} - \|u_f\|^2 + \frac{2f(u_f)}{\|f\|} \\ &= \|u\|^2 - 2\frac{f(u)}{\|f\|} + \|u_f\|^2 . \end{aligned}$$

By (2.2) we have that

$$\frac{-2f(u)}{\|f\|} \ge -2(x, u_f)_s$$

and then

$$\begin{aligned} F_f(u) - F_f(u_f) &\ge \|u\|^2 - 2(x, u_f)_s + \|u_f\|^2 \\ &\ge \|u\|^2 - 2\,\|u\| \cdot \|u_f\| + \|u_f\|^2 \\ &= (\|u\| - \|u_f\|)^2 \ge 0 \end{aligned}$$

which shows that u_f minimizes F_f.
"(ii)⇒ (i)" a). Let $f \in X^* \backslash \{0\}$ and u_f be an element minimizing F_f. Then, for all $\lambda \in \mathbb{R}$ and $u \in X$, we have:

$$F_f(u + \lambda u_f) \geq F_f\left(u_f\right). \tag{2.3}$$

However,

$$\begin{aligned} F_f(u+\lambda u_f) - F_f\left(u_f\right) &= \|u+\lambda u_f\|^2 - \frac{2f(u+\lambda u_f)}{\|f\|} - \|u_f\|^2 + \frac{2f\left(u_f\right)}{\|f\|} \\ &= \|u+\lambda u_f\|^2 - \|u_f\|^2 - \frac{2\lambda f\left(u\right)}{\|f\|} \end{aligned}$$

and then (2.3) is equivalent to

$$\frac{2\lambda f\left(u\right)}{\|f\|} \leq \|u+\lambda u_f\|^2 - \|u_f\|^2 \text{ for all } \lambda \in \mathbb{R} \text{ and } u \in X.$$

Assume that $\lambda > 0$. Then

$$f\left(u\right) \leq \left[\frac{(\|u+\lambda u_f\|^2 - \|u_f\|^2)}{2\lambda}\right] \cdot \|f\|.$$

Letting $\lambda \to 0+$, we get

$$f\left(u\right) \leq \|f\| \left(u, u_f\right)_s$$

for all $u \in X$. Now, changing u with $(-u)$, we get from the previous inequality that

$$f\left(u\right) \geq -\|f\| \left(-u, u_f\right)_s = \|f\| \left(u, u_f\right)_i$$

and then we get the estimation

$$\|f\| \left(u, u_f\right)_i \leq f\left(u\right) \leq \|f\| \left(u, u_f\right)_s \text{ for all } u \in X.$$

Choosing $u = u_f$ we get $f\left(u_f\right) = \|f\|$ and by James' theorem it follows that $(X, \|\cdot\|)$ is reflexive.
"(i)⇒(ii)" b). Assume that there exists a nonzero functional $f_0 \in X^*$ for which we can find at least two distinct vectors

$$u^i_{f_0} (i = 1, 2), \|u^i_{f_0}\| = 1$$

which minimize F_{f_0}. As above (see "(ii)⇒(i)" a).), we get that $f_0(\ u^i_{f_0}) = \|f_0\|$, which, by Krein's theorem contradicts the strict convexity of X.
"(ii)⇒(i)" b). Assume that X is not reflexive. Thus, by Krein's theorem, there exists a continuous linear functional $f_0 \neq 0$ and at least two distinct elements

$$u^i_{f_o} (i = 1, 2), \|u^i_{f_o}\| = 1$$

such that $f_0(\ u^i_{f_o}) = \|f_0\|$. Now, by a similar argument as in "(i)⇒ (ii)" a)., we deduce that $u^i_{f_o}$ $(i = 1, 2)$ will minimize the quadratic functional F_{f_0}, which is a contradiction. ∎

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