Individual Exponetial Stability for Evolution Families of Linear and Bounded Operators

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†

Abstract

Let $\mathbf{R}_+$ be the set of all non-negative real numbers, $I \in \{\mathbf{R}, \mathbf{R}_+\}$ and $\mathcal{U} = \{U(t,s) : t \geq s \in I\}$ be a strongly continuous and exponentially bounded evolution family of bounded linear operators acting on a Banach space X. Let E be a normed function space over I satisfying some properties, see section 2. We prove that if $\chi_I(\cdot)||U(\cdot,s)x||$ defines an element of the space E for some $s \in I$ and some $x \in X$, then there exists $N(s,x) > 0$ such that

$$||U(t,s)|| \leq N(s,x)e^{-\nu(t-s)}, \quad \forall t \geq s.$$

Some related results for periodic evolution families are also proved.

1. Introduction

Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a Banach space X, and $\omega(\mathbf{T})$ be its growth bound. It is well known theorem of Datko [Da1], that if the function $t \mapsto ||T(t)x||$ belongs to $L^2(\mathbf{R}_+)$ for all $x \in X$, then $\omega(\mathbf{T})$ is negative. This result was generalized by Pazy [Pa], who showed that the exponent $p = 2$, may be replaced by $1 \leq p < \infty$, and by Datko [Da2], who showed the following result: *Let* $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$ *be a strongly continuous and*

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exponentially bounded evolution family of bounded linear operators acting on X*, see definitions below. Let us consider the function*

$$t \mapsto U_s^x(t) := \chi_{[s,\infty)}(t)||U(t,s)x|| : \mathbf{R}_+ \to \mathbf{R}_+, \quad s \geq 0, x \in X.$$

If there exists $1 \leq p < \infty$ *such that* U_s^x *belongs to* $L^p(\mathbf{R}_+)$ *for all* $s \geq 0$ *and* $x \in X$ *and if in addition*

$$\sup_{s\geq 0} ||U_s^x||_p = M(x) < \infty \quad \forall x \in X,$$

then the family $\mathcal{U}$ *is uniformly exponentially stable, that is, there exist the constants* $N > 0$ *and* $\nu > 0$ *such that*

$$||U(t,s)|| \leq Ne^{-\nu(t-s)}, \quad \forall t \geq s \geq 0. \tag{1}$$

Recently in [Bus1, Theorem 0.1] it is shown that the evolution family $\mathcal{U}$, as above, is uniformly exponentially stable if for any $x \in X$ and every $s \geq 0$, the map U_s^x belongs to E and

$$\sup_{s\geq 0} |U_s^x|_E = M(x) < \infty.$$

Here E is a normed function space over $\mathbf{R}_+$ with $\Psi_E(\infty) = \infty$, see section 2 for the exact definitions in the present context, which satisfies the condition

$$(\mathbf{C1}) \quad |\chi_{[0,t]}|_E \leq |\chi_{[\tau,t+\tau]}|_E \quad \forall t \geq 0, \forall \tau \geq 0.$$

χ_A is the characteristic function of the set A. For example in $L^p(\mathbf{R}_+, \mathbf{C})$ the condition (**C1**) it is verified with equality. The hypothesis (**C1**) is essential, see [Bus1, Example 3.2], but it may be except in the autonomous case, see [vN, Theorem 4.2]. In this paper we prove that the condition (**C1**) may be except also in the case when $\mathcal{U}$ is a q-periodic evolution family of linear and bounded operators on X. Recall that an evolution family $\mathcal{U}$, as above, is exponentially stable if there exist $\nu > 0$ and a function $N : [0,\infty) \to (0,\infty)$ such that

$$||U(t,s)|| \leq N(s)e^{-\nu(t-s)}, \quad \forall t \geq s \geq 0. \tag{2}$$

In [Bus1, Theorem 3.2] it is shown that $\mathcal{U}$ is exponentially stable if the function U_s^x belongs to E for all $s \geq 0$ and all $x \in X$. Here E is a normed function space satisfying the hypothesis (H), see section 2. The individual variant of this result we will prove in the pesent paper.

2. Definitions. Preliminary results

Let X be a real or complex Banach space. We denote by $L(X)$ the Banach space of all linear operators acting on X. We also denote by $||\cdot||$, the norms of vectors and operators in X and $L(X)$, respectively.

A family $\mathcal{U} := \{U(t,s) : t \geq s \in I\}$ is called strongly continuous evolution family of bounded linear operators on X iff:
($\mathbf{e_1}$) $U(t,s)U(s,r) = U(t,r)$ and $U(t,t) = Id$ for all $t \geq r \geq s \in I$; Id is the identity operator in $L(X)$, and
(e_2) the function $(t,s) \mapsto U(t,s)x$ is continuous for all $x \in X$.
If, in addition, there are $M \geq 1$ and $\omega > 0$ such that
(e_3) $||U(t,s)|| \leq Me^{\omega(t-s)}$ for all $t \geq s \in I$ then $\mathcal{U}$ is called strongly continuous evolution and exponential bounded family on X. Let $q > 0$. A strongly continuous evolution family $\mathcal{U}$ on X would be called q-periodic evolution family on X, if
($\mathbf{e_4}$) $U(t+q, s+q) = U(t,s)$ for all $t \geq s \in I$.
It is easy to show that a q-periodic evolution family on X is an exponentially bounded evolution family on X, see section 4.

Let $(I, \mathcal{L}, m)$ the Lebesgue's measure space, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$ and $\mathcal{M}(I)$ the linear space of all measurable functions $f : I \to \mathbf{K}$, identifiyng functions which are equal a.e. on I. Consider a function $\rho : \mathcal{M}(I) \to [0, \infty]$ with the following properties:
(**n1**) $\rho(f) = 0$ if and only if $f = 0$;
(**n2**) $\rho(af) = |a|\rho(f)$ for any scalar $a \in \mathbf{K}$ and any $f \in \mathcal{M}(I)$, with $\rho(f) < \infty$;
(**n3**) $\rho(f+g) \leq \rho(f) + \rho(g)$ for all $f, g \in \mathcal{M}(I)$.
Let $F = F_\rho$ be the set of all $f \in \mathcal{M}(I)$ such that $|f|_F := \rho(f) < \infty$. It is clear that $(F, |\cdot|)$ is a normed linear space. The normed linear subspace E of F is called *normed function space over I* if the following two conditions hold:
(**n4**) if $f \in E, g \in E$ and $|f| \leq |g|$ a.e., then $|f|_E \leq |g|_E$;
(**n5**) $\chi_{[0,t]} \in E$ for all $t > 0$.

A normed function space E over I has the ideal property if for all $f \in \mathcal{M}(I)$ and any $g \in E$, from $|f| \leq |g|$ a.e. it follows that $f \in E$. It is clear that F_ρ has the ideal property.

In applications of systems theory it is important to consider the shift operator $S_\tau : \mathcal{M}(I) \to \mathcal{M}(I)$, defined for $\tau \in I$ and $t \in \mathbf{R}$, by

$$(S_\tau f)(t) := \begin{cases} 0, & t-\tau \notin I \quad \text{a.e.} \\ f(t-\tau), & t-\tau \in I \quad \text{a.e.} \end{cases}$$

Let E be a normed function space over I. We say that E satisfies the hypothesis (H) if the following conditions hold:
$(\mathbf{h_1})$ $S_\tau f$ belongs to E for all $\tau \in I$ and every $f \in E$, that is the space E is shift invariant;
$(\mathbf{h_2})$ $S_\tau \in L(E)$ for all $\tau \in I$;
$(\mathbf{h_3})$ $|S_\tau f|_E = ||S_\tau||_{L(E)} |f|_E$ for all $f \in E$ and every $\tau \in I$.

Examples: 1. The spaces $L^p(I, \mathbf{K})$ are normed function spaces with the ideal property which satisfies the hypothesis (H) for all $1 \leq p \leq \infty$;

2. Let $a \in \mathbf{R}, 1 \leq p < \infty$ and $exp_\alpha(t) := e^{\alpha t}$ for $\alpha \in \mathbf{R}$ and $t \in \mathbf{R}$. The space

$$L_{ap} = L_{ap}(I, \mathbf{K}) := \{f \in \mathcal{M}(I) : \rho(f) := |f|_{L_{ap}} = (\int_I exp_{ap}(t)|f(t)|^p dt)^{\frac{1}{p}} < \infty\}$$

is a normed function space over I which has the ideal property and satisfies the hypothesis (H). Moreover, it is easy to see that $|S_\tau f|_{L_{ap}} = e^{a\tau}|f|_{L_{ap}}$ for all $\tau \in I$ and every $f \in L_{ap}(I, \mathbf{K})$.

3. The space

$$M_a^p = M_a^p(I, K) := \{f \in \mathcal{M}(I) : |f|_{M_a^p} := \sup_{s \in I} (\int_s^{s+1} exp_{ap}(t)|f(t)|^p dt)^{\frac{1}{p}} < \infty\}$$

is a normed function space over I which has the ideal property and in addition

$$|S_\tau f|_{M_a^p} = e^{a\tau}|f|_{M_a^p}, \quad \forall f \in M_a^p, \forall \tau \in I.$$

Remarks. Let E be a normed function space over I satisfying the hypothesis (H). We consider the function

$$\tau \mapsto g(\tau) := ||S_\tau||_{L(E)} : I \to \mathbf{R}_+.$$

Then:
1. If $I = \mathbf{R}$ then there exists a real number $a = a(E)$ such that $g = exp_a$;
2. If $I = \mathbf{R}_+$ and there exists $t_0 \geq 0$ such that the function g is continuous

in t_0 then there exists $a = a(E) \in \mathbf{R}$ such that $g = exp_a$.
3. If $a = a(E) > 0$ and $0 < b \leq a(E)$ then the space $E_b := exp_b E$ endowed with the norm

$$|f|_{E_b} := |exp_{-b} f|_E, \quad \forall f \in E_b$$

satisfies the hypothesis (H) and $a(E_b) = a(E) - b$.

Proof. 1 and 2. If $f \in E \setminus \{0\}$ then $S_\tau f \in E \setminus \{0\}$ for all $\tau \in I$. Moreover

$$\frac{|S_{u+v} f|_E}{|f|_E} = \frac{|S_u(S_v f)|_E}{|S_v f|_E} \cdot \frac{|S_v f|_E}{|f|_E}, \quad \forall u, v \in I,$$

that is $g(0) = 1$ and $g(u+v) = g(u)g(v)$ for all $u, v \in I$. Now it is easy to see that there exists $a = a(E) \in \mathbf{R}$ such that $g = exp_a$.
3. Let $f \in E_b, \tau \in I$ and $t \geq \tau$. Then:

$$(exp_{-b} S_\tau f)(t) = exp_{-b}(\tau) exp_{-b}(t - \tau) f(t - \tau) = \exp_{-b}(\tau)(S_\tau(exp_{-b} f))(t),$$

that is $exp_{-b} S_\tau f = exp_{-b}(\tau) S_\tau(exp_{-b} f)$ and the desire result is easily follows.

Let E be a normed function space. For all $t > 0$, we define $\Psi_E(t) := |\chi_{[0,t]}|_E$ and $\Psi_E(\infty) = \lim_{t \to \infty} \Psi_E(t)$. With $\sigma(T)$ we denote the *spectrum* of an operator $T \in L(X)$ and with $r(T)$ we denote the *spectral radius* of T. We recall that

$$r(T) = \lim_{n \to \infty} ||T^n||^{\frac{1}{n}} = \sup\{|z| : z \in \sigma(T)\}.$$

3. Individual exponential stability

Theorem 3.1 *Let E be a normed function space satisfying the hypothesis (H) and such that $a(E) > 0$. Let $x \in X$ and $s \in I$, fixed, and $\mathcal{U} = \{U(t,s) : t \geq s \in I\}$ be a strongly continuous and exponentially bounded evolution family on X. If U_s^x belongs to E then there exists $b > 0$ and $N(s,x) > 0$ such that*

$$||U(t,s)x|| \leq N(s,x) e^{-b(t-s)}, \quad \forall t \geq s. \tag{3}$$

Proof. The proof of this theorem is essentially contained in [Bus1, Theorem 3.1, Theorem 3.2]. For reasons of self-containedness and because this is easily different, we give the proof in detail. Let $0 < b < a(E)$ and $E_b := exp_b E$. Let

$$\mathcal{U}_b := \{exp_b(t-s) U(t,s) : \quad t \geq s \in I\}.$$

It is clear that $\mathcal{U}_b$ is a strongly continuous and exponentially bounded evolution family on X and $(U_b)_s^x$ belongs to E_b. We will prove that

$$\lim_{t\to\infty}(U_b)_s^x(t) = \lim_{t\to\infty}||U_b(t,s)x|| = 0.$$

If we suppose the contrary, then there exist $\delta > 0$ and a sequence (t_n) with $t_{n+1} > t_n + 1 > t_n \geq s$ for all $n \geq N$, where $N \in \mathbf{N}$ is sufficiently of large, such that

$$\delta \leq ||U_b(t_n,s)x|| \leq Me^{\omega}||U(t,s)x||, \quad \forall t \in I_n = [t_n - 1, t_n], \forall n \geq N.$$

It follows

$$(U_b)_s^x \geq \frac{\delta}{Me^{\omega}}\chi_{I_n}, \quad \forall n \geq N. \tag{4}$$

Let $S = \exp_{a(E)} E = exp_{a(E)-b}E_b$. Then $|\chi_{I_n}|_{E_b} = |exp_{a(E)-b}\chi_{I_n}|_S$. But

$$exp_{a(E)-b}(t)\chi_{I_n}(t) \geq exp_{a(E)-b}(t_n - 1)\chi_{I_n}(t), \quad \forall t \in I,$$

hence

$$|\exp_{a(E)-b}\chi_{I_n}|_S \geq exp_{a(E)-b}(t_n-1)|\chi_{[0,1]}|_S \quad \forall n \geq N.$$

Now, from (4) it follows

$$|(U_b)_s^x|_{E_b} \geq \frac{\delta}{Me^{\omega}} exp_{a(E)-b}(t_n-1)|\chi_{[0,1]}|_S, \quad \forall t \geq N,$$

which is a contradiction with the fact that $(U_b)_s^x$ belongs to E_b. Now, it is clear that $||U_b(t,s)x|| \leq N(s,x)$ for all $t \geq s$, so (3) holds.

Corollary 3.2 *Let $A(t) \in L(X)$ for all $t \in I$ such that the function $t \mapsto A(t)$ is continuous and bounded on I. Let $s \in I$ and $x \in X$. We denote by $u(t,s,x)$ the solution of the Cauchy problem*

$$\dot{u}(t) = A(t)u(t), \quad u(s) = x.$$

If the function $t \mapsto ||u(t,s,x)||$ belongs to a normed function space E satisfying the hypothesis (H) and $a(E) > 0$, then there is $N(s,x) > 0$ and $b > 0$ such that

$$||u(t,s,x)|| \leq N(s,x)e^{-b(t-s)}, \quad \forall t \geq s.$$

Proof. It follows by Theorem 3.1 for $U(t,\tau) := P(t)P^{-1}(\tau)$, where $P(\cdot)$ is the solution of the operatorial Cauchy problem

$$\dot{U}(t) = A(t)U(t), \quad U(0) = Id.$$

Corollary 3.3 *Let $\mathcal{U}$ and E as in Theorem 3.1. If U_τ^x belongs to E for all $\tau \in I$ and every $x \in X$ then there exist $b > 0$ and a function $N : \mathbf{R}_+ \to (0,\infty)$ such that (2) holds, i.e. $\mathcal{U}$ is exponentially stable.*

Proof. It follows by Theorem 3.1 and uniform boundedness principle.

Corollary 3.4 *Let $\mathcal{U}$ be a q-periodic evolution family on X and E as in Theorem 3.1. If U_0^x belongs to E for all $x \in X$ then $\mathcal{U}$ is uniformly exponentially stable, that is (1) holds.*

Proof. It follows by Corollary 3.3 and the fact that a exponentially stable q-periodic evolution family $\mathcal{U}$ is uniformly exponentially stable, see [Bus2].

In the following section we prove that if *$\mathcal{U}$ is as in Corollary 3.4, E is a normed function space with $\Psi_E(\infty) = \infty$, and U_0^x belongs to E for every $x \in X$, then $\mathcal{U}$ is uniformly exponentially stable.* This result is an improvement of Corollary 3.4, because if E is a normed function space satisfying the hypothesis (H) and $a(E) > 0$ then $\Psi_E(\infty) = \infty$.

With previous notations we suppose that for a fixed $x \in X$ the functions $t \mapsto ||u(t,\tau,x)||$ belongs to E for every $\tau \in I$. We ask if can choose a constant $N(x)$, independent of τ, such that

$$||u(t,\tau,x)|| \leq N(x)e^{-b(t-\tau)}, \quad \forall t \geq \tau. \tag{5}$$

The following example shows that a such selection is not possible.

Example 3.5 *Let $a(t) := \sqrt{2} - \sin\ln(t+1) - \cos\ln(t+1), t \geq 0$. The solution $u(t,\tau,x)$ of the Cauchy problem*

$$\dot{u}(t) = -a(t)u(t), \quad u(\tau) = x, t \geq \tau \geq 0, x \in \mathbf{K}$$

cannot satisfies the relation (5), but there exists $b > 0$ and $N(\tau,x) > 0$ such that

$$||u(t,\tau,x)|| \leq N(\tau,x)e^{-b(t-s)}, \quad \forall t \geq \tau.$$

In fact in this case

$$u(t,\tau,x) = e^{h(\tau)-h(t)}x \text{ and } h(t) = (t+1)(\sqrt{2} - \sin\ln(t+1)).$$

See [Bus1, Example 2.2] or [BG] for details.

4. The periodic case

In this section we will only consider the case $I = \mathbf{R}_+$. We shall prove several lemmas which would be used later.

Lemma 4.1 *A q-periodic evolution family $\mathcal{U}$ on X is exponentially bounded.*

Proof. The function $t \mapsto U(t,0)x$ is continuous on $[0,q]$ for any $x \in X$, so there exists $M(x) > 1$ such that $||U(t,0)x|| \leq M(x)$ for all $t \in [0,q]$. Hence, using uniform boundedness principle it results that there exists $M_q > 1$ such that $||U(t,0)|| \leq M_q$ for any $t \in [0,q]$. Let $\xi = mq + s$ with $m \in \{1,2,\cdots\}$ and $s \in [0,q)$. Then

$$\begin{aligned} ||U(\xi,0)|| &\leq ||U(\xi,mq)||||U(mq,(m-1)q)||\cdots||U(q,0)|| \\ &\leq M_q^{m+1} = M_q e^{\frac{1}{q}(mq)\ln M_q} \leq M_q e^{\frac{1}{q}(\ln M_q)\xi}. \end{aligned}$$

Now, it is not hard to see, that there are $\omega = \omega(q) > 0$ and $M_q > 1$ such that

$$||U(\xi,0)|| \leq M_q e^{\omega\xi}, \quad \forall \xi \geq 0. \tag{3}$$

On the other hand, the function $(t,s) \mapsto U(t,s)x$ is continuous on the set $\{(t,s) \in \mathbf{R}^2 : 0 \leq s \leq t \leq q\}$ for any $x \in X$, so there exists $N_q > 1$ such that

$$||U(t,s)|| \leq N_q, \quad \forall (t,s) \in [0,q] \times [0,q], t \geq s. \tag{4}$$

Let $v \geq \xi + q$ and $\xi = nq + u$, with $n \in \mathbf{N}$ and $u \in [0,q)$. Then from (3) and (4), it follows:

$$\begin{aligned} ||U(v,\xi)|| &\leq ||U(v,(n+1)q)U((n+1)q,\xi)|| \\ &\leq ||U(v-nq-q,0)||||U(q,u)|| \leq M_q N_q e^{\omega(v-\xi)}. \end{aligned}$$

Let $V = U(q,0) \in L(X)$.

Lemma 4.2 *A q-periodic evolution family $\mathcal{U}$ is uniformly exponentially stable if and only if $r(V) < 1$.*

Proof. Let $\nu > 0$ such that $r(V) < e^{-\nu} < 1$. Then $r(e^{\nu}V) < 1$, so there is $K > 0$ such that $||e^{\nu n}V^n|| \leq K$, for any $n = 1,2,\cdots$. It follows $||U(nq,0)|| \leq Ke^{-\nu n}$. Let $t \geq q, m = 1,2,\cdots$ and $\rho \in [0,q)$ such that $t = mq + \rho$. Then

$$\begin{aligned} ||U(t,0)|| &\leq ||U(t,mq)||||U(mq,0)|| \\ &\leq Me^{\omega q}||U(mq,0)|| \leq K_1 e^{-\nu_1 t}, \end{aligned}$$

where $\nu_1 = \frac{\nu}{q}$, and $K_1 = MKe^{\omega q}e^{\frac{\nu\rho}{q}}$.

Let $t \in [0, q)$. Then:

$$||U(t,0)|| \leq K_2 e^{-\nu_1 t}, \text{ where } K_2 = Me^{(\omega+\nu_1)q}.$$

Let $\xi \geq 0$ and $t \geq \xi + q$. It is clear that there are $p \in \mathbf{N}$ and $s \in [0, q)$, such that $\xi = pq + s$. Then:

$$\begin{aligned} ||U(t,\xi)|| &\leq ||U(t,(p+1)q)||||U((p+1)q,s)|| \\ &\leq Me^{\omega q}(K_1 + K_2)e^{-\nu_1(t-\xi)}. \end{aligned}$$

The converse implication is obvious. The proof of Lemma 4.2 is finished.

We present without proof a result in the theory of linear and bounded operators. We will denote by $\mathbf{c_0}$ the Banach space of all sequences (a_n) of complex numbers which are convergent to 0, endowed with the norm $||(a_n)|| = \sup_n |a_n|$.

Lemma 4.3 *Let $T \in L(X)$ with $r(T) \geq 1$ and $\varepsilon \in (0,1)$. For all sequences $(a_n) \in \mathbf{c_0}$ with $||(a_n)|| = 1$, there exists $x_0 \in X$ with $||x_0|| = 1$, such that:*

$$||T^n x_0|| \geq \varepsilon |a_n|, \quad n \in \mathbf{N}.$$

The result contained in Lemma 4.3, was been proved by V. Müller in [M] .

Lemma 4.4 *Let $\mathcal{U}$ be a q-periodic evolution family on the Banach space X and $V = U(q,0)$. If $r(V) \geq 1$ then there is $C > 0$ such that for all $\gamma \in C_0(\mathbf{R}_+, \mathbf{C})$, (the space of all continuous functions with $\lim_{t\to\infty} \gamma(t) = 0$), of norm one, there exists $x_0 \in X$ with $||x_0|| = 1$, such that*

$$||U(t,0)x_0|| \geq C|\gamma(t)|, \quad \forall t \geq 0.$$

Proof. Let $\alpha \in C_0(\mathbf{R}_+)$ be a decreasing function of norm one such that $\alpha(s) \geq |\gamma(s)|$ for all $s \geq 0$ and let $\beta \in C_0(\mathbf{R}_+)$, the function defined by

$$\beta(t) = \begin{cases} \alpha(0), & 0 \leq t < q \\ \alpha(t-q), & t \geq q. \end{cases}$$

Let $t > q, k \in \mathbf{N}$ and $r \in [0, q)$ such that $t = kq + r$. From Lemma 3 it follows that for some $x_0 \in X$, $||x_0|| = 1$ we have:

$$\begin{aligned} \tfrac{1}{2}|\gamma(t)| &\leq \tfrac{1}{2}\alpha(t) \leq \tfrac{1}{2}\alpha(kq) = \tfrac{1}{2}\beta((k+1)q) \\ &\leq ||V^{k+1}x_0|| = ||U((k+1)q,0)x_0|| \\ &\leq Me^{\omega q}||U(t,0)x_0||. \end{aligned}$$

The following result it is a generalization of Datko-Pazy's theorem.

Theorem 4.5 *Let $\mathcal{U}$ be a q-periodic evolution family on the Banach space X and $V = U(q,0)$. Let also E be a normed function space over $\mathbf{R}_+$, which has the ideal property and such that $\Psi_E(\infty) = \infty$. If for every $x \in X$, the function $t \mapsto ||U(t,0)x||$ belongs to E, then $r(V) < 1$, i.e. $\mathcal{U}$ is uniformly exponentially stable.*

Proof. Firstly it is proves by induction that there exists a strictly increasing sequence (t_n) such that

$$|\chi_{[t_n,t_{n+1})}|_E \geq (n+1)^2, \quad n = 0, 1, \cdots .$$

This follows because $\Psi_E(\infty) = \infty$. Suppose that $r(V) \geq 1$. Let $\gamma \in C_0(\mathbf{R}_+)$ such that $\gamma(t) \geq \frac{1}{n}$ for any $t \in [t_{n-1}, t_n)$ and any $n = 1, 2, \cdots$. Let C and x_0 as in Lema 4. Then we have:

$$\begin{aligned} |||U(\cdot,0)x_0|||_E &\geq |\chi_{[t_{n-1},t_n)}||U(\cdot,0)x_0|||_E \\ &\geq C|\chi_{[t_{n-1},t_n)}\gamma|_E \\ &\geq \frac{C}{n}|\chi_{[t_{n-1},t_n)}|_E \geq Cn \end{aligned}$$

for all $n = 1, 2, \cdots$. It follows $|||U(\cdot,0)x_0|||_E = \infty$. This contradiction concludes the proof.

Remark If X is a complex Banach space then Theorem can be deduced directly from the following result of van Neerven (see [vN, Theorem 2.3 and the Proof of Theorem 4.2]):

Let F be a Banach function space over $\mathbf{N}$ such that $\Psi_F(\infty) = \infty$, and let $T \in L(X)$. If $(||T^n(x)||) \in F$ for all $x \in X$, then $r(V) < 1$.

Indeed, the space

$$F := \{(a_n) : \sum_{n \geq 0} a_n \chi_{[n,n+1]} \in E\}$$

is a Banach function space over $\mathbf{N}$ such that $\Psi_F(\infty) = \infty$. Let $V = U(q,0)$. The exponential boundedness of the evolution family $\mathcal{U}$ and the identity

$$V^{n+1} = U(n+1,0) = U(n+1,t)U(t,0), \quad \forall t \in [n, n+1]$$

then imply

$$||V^{n+1}x|| \leq C||U(t,0)x||$$

for a constant C independent of $n \in \mathbf{N}$ and $t \in [n, n+1]$. Thus, by previous result $r(V) < 1$. Hence $\mathcal{U}$ is uniformly exponentially stable.

Theorem 4.6 *Let $\mathcal{U}$ be a q-periodic evolution family on the Banach space X and E_1, E_2 be two normed function spaces over $\mathbf{R}_+$. If E_1 has the ideal property and $\Psi_{E_2}(\infty) = \infty$ and if $\mathcal{U}$ is (E_1, E_2)-stable (i.e. the function*

$$t \mapsto || \int_0^t U(t,\xi) f(\xi) d\xi|| \text{ belongs to } E_2 \text{ for all } f \in L^1_{loc}(\mathbf{R}_+, X) \text{ with}$$

$$||f(\cdot)|| \in E_1),$$

then $\mathcal{U}$ is exponentially stable.

Proof. Let $h > 0, x \in X$ and $f = \chi_{[0,h]}(\cdot)U(\cdot, 0)x$. It is clear that

$$||f(t)|| \leq M e^{\omega h} \chi_{[0,h]}(t) ||x|| \quad \forall x \in X,$$

so $||f(\cdot)|| \in E_1$. Moreover,

$$\int_0^t U(t,\xi) f(\xi) d\xi = hU(t,0)x, \quad \forall t \geq h.$$

Then:

$$||U(t,0)x|| \leq h^{-1} || \int_0^t U(t,\xi) f(\xi) d\xi|| + M e^{\omega h} \chi_{[0,h]}(t) ||x||, \quad \forall t \geq 0$$

so, $||U(\cdot, 0)x|| \in E_2$. We apply Theorem 4.5 and it results that $\mathcal{U}$ is uniformly exponentially stable.

Remark. Let $1 \leq p < \infty$ and $\mathbf{T}_p = \{T_p(t)\}_{t \geq 0}$ the evolution semigroup associated to the q-periodic evolution family $\mathcal{U}$, on the usual Lebesgue-Bochner space $L^p(\mathbf{R}_+, X)$. Precisely, $T_p(t)$ is given by

$$(T_p(t)f(s) = \begin{cases} U(s, s-t) f(s-t), & s \geq t \\ U(s,0) f(0), & 0 \leq s \leq t. \end{cases}$$

For more details about evolution semigroups and their connection with asymptotic properties of the evolution families, we refer to [LRS], [MRS], [RS1],

[RS2], and the literatur cited therein. Using Theorem 4.6 it can be proved, as in [MRS, Corollary 2.4], the spectral mapping theorem for $\mathbf{T_p}$, that is:

$$e^{t\sigma(G_p)} = \sigma(T_p(t)) \setminus \{0\}, \quad \forall t \geq 0$$

and

$$\sigma(G_p) = \{\lambda \in \mathbf{C} : \text{ Re } (\lambda) \leq s(G_p)\}$$

where G_p is the infinitesimal generator of $\mathbf{T}_p$ and $s(G_p) = \sup\{ \text{ Re } (z) : z \in \sigma(G_p)\}$.

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