ON A BOJANIĆ–STANOJEVIĆ TYPE INEQUALITY AND ITS APPLICATIONS

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Abstract. An extension of the Bojanić–Stanojević type inequality [1] is made, considering the r-th derivate of the Dirichlets's kernel $D_k^{(r)}$ instead of D_k.

Namely, the following inequality is proved:

$$\Big\|\sum_{k=1}^{n}\alpha_k D_k^{(r)}(x)\Big\|_1 \le M_p n^{r+1}\Big(\frac{1}{n}\sum_{k=1}^{n}|\alpha_k|^p\Big)^{1/p},$$

where $\|\cdot\|_1$ is the L^1-norm, $\{\alpha_k\}$ is a sequence of real numbers, $1 < p \le 2$, $r = 0, 1, 2, \ldots$ and M_p is an absolute constant dependents only on p. As an application of this inequality, it is shown that the class $\mathcal{F}_{pr}$ is a subclass of $\mathcal{BV}\cap\mathcal{C}_r$, where $\mathcal{F}_{pr}$ is the extension of the Fomin's class, $\mathcal{C}_r$ is the extension of the Garrett–Stanojević class [7] and $\mathcal{BV}$ is the class of all null sequences of bounded variation.

1. Introduction.

Sidon [5] proved the inequality named after him in 1939 year. It is an upper estimate for the integral norm of a linear combination of trigonometric Dirichlet kernels expressed in terms of the coefficients. Since the estimate has many applications for instance in L^1-convergence problems and summation methods with respects to trigonometric series, newer and newer improvements of the original inequality has been proved by several authors.

Fomin [3] applying the linear method for summing of Fourier series has given another proof of this inequality. Thus the inequality is called as Sidon-Fomin's inequality. Also, S. A. Telyakovskii in [6] has given an elegant proof of Sidon-Fomin's inequality.

Lemma 1 (Sidon-Fomin). Let $\{\alpha_k\}_{k=0}^n$ be a sequence of real numbers such that $|\alpha_k| \le 1$ for all k. Then there exists a positive constant M such that for any $n \ge 1$,

$$\Big\|\sum_{k=0}^{n}\alpha_k D_k(x)\Big\|_1 \le M(n+1)\,. \tag{1.1}$$

In [8] we extended this result and we given two different proof of the following lemma.

Lemma 2 [8]. Let $\{\alpha_j\}_{j=0}^k$ be a sequence of real numbers such that $|\alpha_k| \le 1$ for all k. Then there exists a positive constant $M > 0$, such that for any $n \ge 1$,

$$\Big\|\sum_{k=0}^{n}\alpha_k D_k^{(r)}(x)\Big\|_1 \le M(n+1)^{r+1}\,. \tag{1.2}$$

But Bojanić and Stanojević [1] proved the following more general inequality of (1.1).

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Lemma 3 [1]. Let $\{\alpha_i\}_{i=0}^n$ be a sequence of real numbers. Then for any $1 < p \le 2$ and $n \ge 1$

$$\Big\|\sum_{k=0}^n \alpha_k D_k(x)\Big\|_1 \le M_p(n+1)\Big(\frac{1}{n+1}\sum_{k=1}^n |\alpha_k|^p\Big)^{1/p}, \tag{1.3}$$

where the constant M_p dependents only on p.

We note that this estimate is essentially contained (case $p = 2$) in Fomin [3]. Sidon-Fomin's inequality is a special case of the Bojanić-Stanojević inequality, i.e. it can easily be deduced from Lemma 3.

It is easy to see that Bojanić-Stanojević inequality is not valid for $p = 1$. Indeed, if $\alpha_n = 1$ and $\alpha_k = 0$ ($k \ne n$, $k \in \mathbf{N}$) then the left side is of order $\dfrac{\log n}{n}$ while the right side is of order $\dfrac{1}{n}$ as $n \to \infty$.

For the proof of our new results we need the following lemma.

Lemma 4 [9]. If $T_n(x)$ is a trigonometrical polynomial of order n, then

$$\|T_n^{(r)}\| \le n^r \|T_n\|\,.$$

This is S. Bernstein's inequality in the $L^1(0,\pi)$-metric (see [9], Vol.2. p.11).

2. Main result

Now we will prove a counterpart of inequality (1.3) in the case where the r-th derivate of the Dirichlet's kernel $D_k^{(r)}$ is used instead of $D(x)$.

Lemma 5. Let $\{\alpha_k\}_{k=1}^n$ be a sequence of real numbers. Then for any $1 < p \le 2$ and $r = 0, 1, 2, \ldots, n \in \mathbf{N}$ the following inequality holds:

$$\Big\|\sum_{k=1}^n \alpha_k D_k^{(r)}(x)\Big\|_1 \le M_p n^{r+1}\Big(\frac{1}{n}\sum_{k=1}^n |\alpha_k|^p\Big)^{1/p}, \tag{2.1}$$

where the constant M_p dependents only on p.

Proof. Without loss of generality, we may assume that n is of the form $n = 2^m - 1$ with some $m \ge 1$. Let $j \ge 1$. Appliyng first Bernstein's inequality, then Bojanić-Stanojević inequality, yields

$$\Big\|\sum_{k=2^{j-1}}^{2^j-1} \alpha_k D_k^{(r)}\Big\| \le (2^{j-1})^r \Big\|\sum_{k=2^{j-1}}^{2^j-1} \alpha_k D_k\Big\| \le$$

$$\le (2^{j-1})^r M_p 2^{(j-1)(1-1/p)}\Big(\sum_{k=2^{j-1}}^{2^j-1} |\alpha_k|^p\Big)^{1/p}.$$

Continuing by making use of the triangle inequality, then Holder's inequaltiy with the exponents p and q, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we get:

$$\Big\|\sum_{k=1}^{2^m-1} \alpha_k D_k^{(r)}\Big\| \le \sum_{j=1}^m \Big\|\sum_{k=2^{j-1}}^{2^j-1} \alpha_n D_k^{(r)}\Big\|$$

$$\le M_p\Big(\sum_{j=1}^{m}(2^{j-1})^{rq}2^{(j-1)(1-1/p)q}\Big)^{1/q}\Big(\sum_{j=1}^{m}\sum_{k=2^{j-1}}^{2^j-1}|\alpha_k|^p\Big)^{1/p}$$

$$\le M_p(2^{m-1})^r\Big(\sum_{j=1}^{m}2^{j-1}\Big)^{1/q}\Big(\sum_{k=1}^{2^m-1}|\alpha_k|^p\Big)^{1/p}$$

$$\le M_p(2^m-1)^r(2^m-1)^{1/q}\Big(\sum_{k=1}^{2^m-1}|\alpha_k|^p\Big)^{1/p}$$

$$= M_p n^{r+1}\Big(\frac{1}{n}\sum_{k=1}^{n}|\alpha_k|^p\Big)^{1/p}.$$

It is easy to see that the inequality (1.2) is a special case of the inequality (2.1), i.e. it can easily be deduced from Lemma 5.

3. Application

The problem of L^1-convergence, via Fourier coefficient, consists of finding the properties of Fourier coefficients such that the necessary and sufficiant condition for $\|S_n - f\| = o(1)$, $n \to \infty$ is given in the form $a_n \lg n = o(1)$, $n \to \infty$. Here S_n is the partial sums of the cosine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx\,.$$

Sidon-Telyakovskii class $\mathcal{S}$ [6] is a classical example for which condition $a_n \lg n = o(1)$, $n \to \infty$ is equivalent to $\|S_n - f\| = o(1)$, $n \to \infty$ Later Fomin [2] have extended the Sidon-Telyakovskii class. He defined a class $\mathcal{F}_p$, $p > 1$ of Fourier coefficients as follows: a sequence $\{a_k\}$ belongs to $\mathcal{F}_p$, $p > 1$ if $a_k \to 0$ as $k \to \infty$ and

$$\sum_{k=1}^{\infty}\Big(\frac{1}{k}\sum_{i=k}^{\infty}|\Delta a_i|^p\Big)^{1/p} < \infty\,. \tag{3.1}$$

We note that Fomin [2] has given an equivalent form of the condition (3.1). Namely, he proved that $\{a_n\} \in \mathcal{F}_p$, $p > 1$ iff $\sum\limits_{s=1}^{\infty} 2^s\,\Delta_s^{(p)} < \infty$, where

$$\Delta_s^{(p)} = \Big\{\frac{1}{2^{s-1}}\sum_{k=2^{s-1}+1}^{2^s}|\Delta a_k|^p\Big\}^{1/p}.$$

Let $\mathcal{BV}$ denote the class of null sequence $\{a_n\}$ od bounded variation, i.e. $\sum\limits_{n=1}^{\infty}|\Delta a_n| < \infty$.

The class $\mathcal{C}$ was defined by Garrett and Stanojević [4] as follows: a null sequence of real numbers satisfy the condition $\mathcal{C}$ if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ independent of n, such that

$$\int_0^{\delta}\Big|\sum_{k=n}^{\infty}\Delta a_k D_k(x)\Big|dx < \varepsilon\,, \quad \text{for every } n\,.$$

On the other hand, Stanojević [10] proved the following inclusion between the classes $\mathcal{F}_p$, $\mathcal{C}$ and $\mathcal{BV}$.

Theorem 1 [10]. For all $1 < p \le 2$ the following inclusion holds $\mathcal{F}_p \subset \mathcal{BV} \cap \mathcal{C}$.

In [7] we defined an extension $\mathcal{C}_r$, $r = 0, 1, 2, \dots,$ of the Garrett-Stanojević class. Namely, a null sequence $\{a_k\}$ belongs to the class $\mathcal{C}_r$, $r = 0, 1, 2, \dots$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\int_0^\delta \Big| \sum_{k=n}^\infty \Delta a_k D_k^{(r)}(x) \Big| < \varepsilon \,, \quad \text{for all } n\,.$$

When $r = 0$, we denote $\mathcal{C}_r = \mathcal{C}$.

Denote by I_m the dyadic interval $[2^{m-1}, 2^m)$, for $m \ge 1$. A null sequence $\{a_n\}$ belongs to the class $\mathcal{F}_{pr}$, $p > 1$, $r = 0, 1, 2, \dots$ if

$$\sum_{m=1}^\infty 2^{m(1/q+r)} \Big(\sum_{k\in I_m} |\Delta a_k|^p \Big)^{1/p} < \infty, \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1\,.$$

It is obvious that $\mathcal{F}_{pr} \subset \mathcal{F}_p$. For $r = 0$, we obtain the Fomin's class $\mathcal{F}_p$.

Theorem 2. For all $1 < p \le 2$ and $r = 0, 1, 2, \dots$ the following inclusion holds $\mathcal{F}_{pr} \subset \mathcal{BV} \cap \mathcal{C}_r$.

Proof. By theorem 1, it is clear that $\mathcal{F}_{pr} \subset \mathcal{BV}$. It suffices to show that

$$\Big\| \sum_{k=n}^\infty \Delta a_k D_k^{(r)}(x) \Big\| = o(1)\,, \quad n \to \infty\,.$$

Since

$$\sum_{m=1}^\infty 2^{m(1/q+r)} \Big(\sum_{k\in I_m} |\Delta a_k|^p \Big)^{1/p} = 2 \sum_{m=1}^\infty \Big\{ 2^{(m-1)[(r+1)p-1]} \sum_{k\in I_m} |\Delta a_k|^p \Big\}^{1/p} ,$$

we have:

$$\sum_{k=1}^\infty k^{(r+1)p-1} |\Delta a_k|^p < \infty\,.$$

Applying the Lemma 6, we obtain

$$\Big\| \sum_{k=n}^\infty \Delta a_k D_k^{(r)}(x) \Big\| \le M_p \Big(\sum_{k=n}^\infty k^{(r+1)p-1} |\Delta a_k|^p \Big) = o(1), \quad n \to \infty\,.$$

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