TWO NEW L^1–ESTIMATES FOR TRIGONOMETRIC SERIES WITH FOMIN'S COEFFICIENT CONDITION

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Abstract. L^1-estimates will be established in this paper for cosine and sine trigonometric series, considering the Fomin's class $\mathcal{F}_p$, $p > 1$ of Fourier coefficients.

1. Introduction

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1.1}$$

$$g(x) = \sum_{n=1}^{\infty} a_n \sin nx \tag{1.2}$$

be the cosine and sine trigonometric series.

It's well – known (see [1], [2], [3]) that if $\{a_n\}$ is null-quasi-convex sequence of real numbers, the series (1.1) is a Fourier series of some $f \in L^1$ and the following estimate holds:

$$\int_a^\pi |f(x)|dx \leq \frac{\pi}{2} \sum_{k=1}^{\infty} k|\Delta^2 a_{k-1}| \, . \tag{1.3}$$

Let $\mathcal{BV}$ denote the class of sequences of bounded variation. The following two theorems were proved by Telyakovskii [4], [5].

Theorem 1. Let $\{a_n\} \in \mathcal{BV}$, $\sum_{i=2}^{\infty} \left| \sum_{k=1}^{[i/2]} \frac{\Delta a_{i-k} - \Delta a_{i+k}}{k} \right| < \infty$, then the following estimate holds,

$$\int_0^\pi |f(x)|dx \leq C \left(\sum_{k=0}^{\infty} |\Delta a_k| + \sum_{i=2}^{\infty} \left| \sum_{k=1}^{[i/2]} \frac{\Delta a_{i-k} - \Delta a_{i+k}}{k} \right| \right) ,$$

where C is some absolute constant.

Theorem 2. Let $\{a_n\} \in \mathcal{BV}$, $a_0 = 0$,

$$\sum_{i=2}^{\infty} \left| \sum_{k=1}^{[i/2]} \frac{\Delta a_{i-k} - \Delta a_{i+k}}{k} \right| < \infty$$

then the following estimate holds uniformly with respect to $s = 1, 2, 3, \ldots$ for the function g:

$$\left| \int_{\pi/(2s+1)}^{\pi} |g(x)|dx - \sum_{k=1}^{s} \frac{|a_k|}{k} \right| \leq C \left(\sum_{k=0}^{\infty} |\Delta a_k| + \sum_{i=2}^{\infty} \left| \sum_{k=1}^{[i/2]} \frac{\Delta a_{i-k} - \Delta a_{i+k}}{k} \right| \right) ,$$

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where C is some absolute constant.

Recently, Telyakovskii [6] proved the following inequality:

$$\sum_{i=2}^{\infty}\left|\sum_{k=1}^{[i/2]}\frac{\Delta a_{i-k}-\Delta a_{i+k}}{k}\right| \le C\sum_{k=1}^{\infty}k|\Delta^2 a_{k-1}|\,. \tag{1.4}$$

We note that, if a_k is a null quasi-convex sequence, then

$$\sum_{k=0}^{\infty}|\Delta a_k| \le \sum_{k=1}^{\infty}k|\Delta^2 a_{k-1}|\,.$$

Thus the estimate (1.3) follows from the Theorem 1 and the estimate (1.4) with some absolute constant C instead of $\frac{\pi}{2}$. As a consequence of the Theorem 2 and estimate (1.4), we obtain the following Theorem.

Theorem 3. If $\{a_n\}$ is a null-quasi-convex sequence, $a_0 = 0$, then (1.2) will be a Fourier series iff $\sum_{n=1}^{\infty}\frac{|a_n|}{n} < \infty$. Moreover, if $\sum_{n=1}^{\infty}\frac{|a_n|}{n} < \infty$, then the following estimate holds:

$$\left|\int_0^{\pi}\left|\sum_{k=1}^{\infty}a_k\sin kx\right|dx-\sum_{k=1}^{\infty}\frac{|a_k|}{k}\right| \le C\sum_{k=1}^{\infty}k|\Delta^2 a_{k-1}|\,.$$

Also, Telyakovskii [7] has given and direct proof of the Theorem 3, proving the following estimate:

$$\left|\int_{1/s+1}^{\pi}\left|\sum_{k=1}^{\infty}a_k\sin kx\right|dx-\sum_{k=1}^{s}\frac{|a_k|}{k}\right| \le C\sum_{k=1}^{\infty}k|\Delta^2 a_{k-1}|\,.$$

Fomin [8] defined a class $\mathcal{F}_p$, $p>1$ of Fourier coefficients as follows: a sequence $\{a_k\}$ belongs to $\mathcal{F}_p$, $p>1$ if $a_k \to 0$ as $k\to\infty$ and

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=k}^{\infty}|\Delta a_i|^p\right)^{1/p} < \infty\,.$$

In the same paper [8], Fomin proved the following estimate:

$$\sum_{i=2}^{\infty}\left|\sum_{k=1}^{[i/2]}\frac{\Delta a_{i-k}-\Delta a_{i+k}}{k}\right| \le C_p\sum_{s=0}^{\infty}2^s\Delta_s^{(p)}\,, \tag{1.5}$$

where

$$\Delta_s^{(p)} = \left\{\frac{1}{2^{s-1}}\sum_{k=2^{s-1}+1}^{2^s}|\Delta a_k|^p\right\}^{1/p}, \quad \text{for any} \quad 1<p\le 2$$

and any absolute constant C_p dependent or p.

Lemma 1 [9]. If $0 < q < 1$ and $\sum\limits_{n=1}^{\infty} b_n^q < \infty$ $(b_n > 0)$, then the following inequality holds:

$$\left(\frac{q}{1-q}\right)^q \sum_{n=1}^{\infty} b_n^q < \sum_{n=1}^{\infty} \left(\frac{b_n + b_{n+1} + \cdots}{n}\right)^q .$$

2. Main results

Theorem 4. Let $\{a_n\} \in \mathcal{F}_p$, $1 < p \le 2$, then the series (1.1) is a Fourier series and the following inequality holds:

$$\int_0^{\pi} |f(x)|dx \le C_p \sum_{n=1}^{\infty} \left(\frac{\sum\limits_{k=n}^{\infty} |\Delta a_k|^p}{n}\right)^{1/p} .$$

Proof. Putting $b_n = |\Delta a_n|^p$ in Lemma 1, where $q = \dfrac{1}{p}$, we get:

$$\left(\frac{1}{p-1}\right)^{1/p} \sum_{n=1}^{\infty} |\Delta a_n| < \sum_{n=1}^{\infty} \left(\frac{|\Delta a_n|^p + |\Delta a_{n+1}|^p + \cdots}{n}\right)^{1/p}$$

i.e.

$$\sum_{n=1}^{\infty} |\Delta a_n| < (p-1)^{1/p} \sum_{n=1}^{\infty} \left(\frac{\sum\limits_{k=n}^{\infty} |\Delta a_k|^p}{n}\right)^{1/p} .$$

On the other hand, since

$$U_s = \frac{1}{s} \sum_{k=s}^{\infty} |\Delta a_k|^p$$

is monotone decreasing sequence, we obtain

$$\sum_{s=1}^{n} 2^s \Delta s^{(p)} \le 2 \sum_{s=1}^{n} \left[2^{(s-1)(p-1)} \sum_{k=2^{s-1}+1}^{2^s} |\Delta a_k|^p\right]^{1/p}$$

$$\le 2 \sum_{s=1}^{n} 2^{s-1} \left[\frac{1}{2^{s-1}} \sum_{k=2^{s-1}}^{\infty} |\Delta a_k|^p\right]^{1/p}$$

$$= O\left(\sum_{s=1}^{2^{n-1}} (U_s)^{1/p}\right) .$$

Letting $n \to \infty$, we have:

$$\sum_{s=1}^{\infty} 2^s \Delta s^{(p)} = O\left(\sum_{s=1}^{\infty}\left(\frac{1}{s}\sum_{k=s}^{\infty}|\Delta a_k|^p\right)^{1/p}\right).$$

Then applying Theorem 1 and estimation (1.5) the our inequality is satisfied.

Similarly as in the proof of this theorem, applying the Theorem 2, we obtain the following Theorem.

Theorem 5. Let for any $1 < p \le 2$, $\{a_n\} \in \mathcal{F}_p$ and $a_0 = 0$. Then (1.2) will be Fourier series iff $\sum_{n=1}^{\infty}\frac{|a_n|}{n} < \infty$. Moreover if $\sum_{n=1}^{\infty}\frac{|a_n|}{n} < \infty$, then

$$\left|\int_0^{\pi}\left|\sum_{k=1}^{\infty} a_k \sin kx\right| dx - \sum_{k=1}^{\infty}\frac{|a_k|}{k}\right| \le C_p \sum_{n=1}^{\infty}\left(\frac{\sum_{k=n}^{\infty}|\Delta a_k|^p}{n}\right)^{1/p}.$$

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