Weighted inequalities in triangle geometry

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Abstract

The paper contains two parts. In the first we point some applications of a weighted inequality and in the second part the equality conditions are obtained.

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In [1] it is proved the following:

Proposition 1 *Let m, n, p be real numbers such that $m+n>0, n+p>0, p+m>0, mn+np+pm>0$. Then in any triangle ABC the following inequality holds:*

$$ma^2+nb^2+pc^2 \geq 4\sqrt{mn+np+pm}S \tag{1}$$

with standard notations.

Some applications are given in the cited paper:

$$a^2+b^2+c^2 \geq 4\sqrt{3}S \quad \text{for } m=n=p \tag{2}$$

$$a^4+b^4+c^4 \geq 4\sqrt{a^2b^2+b^2c^2+c^2a^2}S \quad \text{for } m=a^2, n=b^2, p=c^2 \tag{3}$$

and therefore:

$$a^4+b^4+c^4 \geq 16S^2 \tag{3'}$$

$$9a^2+5b^2-3c^2 \geq 4\sqrt{3}S \quad \text{for } m=9, n=5, c=-3 \tag{4}$$

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$$(5) \qquad 27a^2 + 27b^2 - 13c^2 \geq 12\sqrt{3}S \quad \text{for } m = n = 27, p = -13$$

$$(6) \qquad 3a^2 - b^2 + 15c^2 \geq 12\sqrt{3}S \quad \text{for } m = 3, n = -1, p = 15.$$

Let us point some other applications of (1):

I) the problem O:553 from Gazeta Matematică, no. 5-6(1988), p. 260 (without author):

$$(7) \qquad 3a^2 + 3b^2 - c^2 \geq 4\sqrt{3}S \quad \text{for } m = n = 3, c = -1.$$

II) the problem E 3150 proposed by George A. Tsintsifas in American Mathematical Monthly, vol. 93(1986), p. 400:

$$(8) \qquad \frac{m}{n+p}a^2 + \frac{n}{p+m}b^2 + \frac{p}{m+n}c^2 \geq 2\sqrt{3}S$$

where m, n, p are positive real numbers. From (1) we have:

$$\frac{m}{n+p}a^2 + \frac{n}{p+m}b^2 + \frac{p}{m+n}c^2 \geq$$

$$\geq 4\sqrt{\frac{mn}{(n+p)(p+m)} + \frac{np}{(p+m)(m+n)} + \frac{pm}{(n+p)(m+n)}}S.$$

Therefore it must be proved that:

$$\frac{mn}{(n+p)(p+m)} + \frac{np}{(p+m)(m+n)} + \frac{pm}{(m+n)(n+p)} \geq \frac{3}{4}$$

or, equivalent:

$$(9) \quad mn(m+n) + np(n+p) + pm(p+m) \geq \frac{3}{4}(m+n)(n+p)(p+m).$$

But the left-hand side of (9) is $m^2n+mn^2+n^2p+np^2+p^2m+pm^2$ and the right-dand side of (9) is $\frac{3}{4}(2mnp + m^2n + mn^2 + p^2m + pm^2 + n^2p + np^2)$. Then (9) is equivalent with $m^2n + mn^2 + n^2p + np^2 + p^2m + pm^2 \geq 6mnp$ which is consequence of AM-GM inequality. For others three solutions of (8) see the cited journal, vol. 95(1988), p. 658-659.

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A natural question with respect to (1) is: when equality holds? The aim of this paper is to give the answer. More precisly, we will show:

Proposition 2 *In* (1) *there is equality if and only if:*

$$\frac{a}{\sqrt{n+p}}=\frac{b}{\sqrt{p+m}}=\frac{c}{\sqrt{m+n}}. \tag{14}$$

Proof From generalized Pitagora's theorem for c and expression $S=\frac{1}{2}ab\sin C$ it results that in (1) is equality if and only if:

$$ma^2+nb^2+p\left(a^2+b^2-2ab\cos C\right)=2\sqrt{mn+\ldots}ab\sin C\Leftrightarrow$$

$$(m+p)\frac{a}{b}+(n+p)\frac{b}{a}=2\left(p\cos C+\sqrt{mn+\ldots}\sin C\right).$$

From AM-GM inequality we have

$$(m+n)\frac{a}{b}+(n+p)\frac{b}{a}\geq 2\sqrt{(m+n)(n+p)}$$

and from Cauchy-Buniakowoski-Schwartz inequality we get

$$\sqrt{(m+n)(n+p)}\geq\left(p\cos C+\sqrt{mn+\ldots}\sin C\right).$$

From last three relations it results that in (1) is equality if and only if $(m+n)\frac{a}{b}=(n+p)\frac{b}{a}$ and $\frac{\cos C}{p}=\frac{\sin C}{\sqrt{mn+\ldots}}$ which means:

$$\frac{a}{\sqrt{n+p}}=\frac{b}{\sqrt{m+p}}\overset{denote}{=}k \tag{15_1}$$

$$\frac{\cos C}{p}=\frac{\sin C}{\sqrt{mn+\ldots}}=\frac{1}{\sqrt{(m+n)(n+p)}}. \tag{15_2}$$

Replaceing $\cos C=\frac{p}{\sqrt{(m+n)(n+p)}}$ from (15_2) and b from (15_1) in generalized Pitagora's theorem we have $c^2=a^2\left(k^2+1\right)-2a^2k\frac{p}{\sqrt{(m+n)(n+p)}}$. But $k=$

$\sqrt{\frac{m+p}{n+p}}$ and then $\left(\frac{c}{a}\right)^2 = 1 + \frac{m+n}{n+p} - 2\sqrt{\frac{m+n}{n+p}}\frac{p}{\sqrt{(m+n)(n+p)}} = \frac{m+n}{n+p}$. Therefore $\frac{a}{\sqrt{n+p}} = \frac{c}{\sqrt{m+n}}$ and this last relation with (15_2) gives the conclusion. □

Consequences:

$$a^2 + b^2 + c^2 = 4\sqrt{3}S \Leftrightarrow a = b = c$$

$$\frac{a^4 + b^4 + c^4}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}} = 4S \Leftrightarrow \frac{a^2}{b^2 + c^2} = \frac{b^2}{c^2 + a^2} = \frac{c^2}{a^2 + b^2} \Leftrightarrow a = b = c$$

$$a^4 + b^4 + c^4 = 16S^2 \Leftrightarrow a = b = c$$

$$9a^2 + 5b^2 - 3c^2 = 4\sqrt{3}S \Leftrightarrow \frac{a}{\sqrt{2}} = \frac{b}{\sqrt{6}} = \frac{c}{\sqrt{14}} \Leftrightarrow a = \frac{b}{\sqrt{3}} = \frac{c}{\sqrt{7}}$$

$$27a^2 + 27b^2 - 13c^2 = 12\sqrt{3}S \Leftrightarrow \frac{a}{\sqrt{14}} = \frac{b}{\sqrt{14}} = \frac{c}{\sqrt{54}} \Leftrightarrow \frac{a}{\sqrt{7}} = \frac{b}{\sqrt{7}} = \frac{c}{\sqrt{27}}$$

$$3a^2 - b^2 + 15c^2 = 12\sqrt{3}S \Leftrightarrow \frac{a}{\sqrt{14}} = \frac{b}{\sqrt{18}} = \frac{c}{\sqrt{2}} \Leftrightarrow \frac{a}{\sqrt{7}} = \frac{b}{3} = c$$

$$3a^2 + 3b^2 - c^2 = 4\sqrt{3}S \Leftrightarrow \frac{a}{\sqrt{2}} = \frac{b}{\sqrt{2}} = \frac{c}{\sqrt{6}} \Leftrightarrow a = b = \frac{c}{\sqrt{3}}.$$

References

[1] Chiriţă, M., Gonciulescu, C., *On some inequalities in a triangle* (Romanian), Matematica pentru elevi, Galaţi, no. 8(1989), 28-29.

[2] Becheanu, M., *Algebraic methods in triangle geometry*, Romanian Mathematics Magazin, no. 1(1999), 25-40.