

SOME INEQUALITIES FOR THE DISPERSION OF A RANDOM VARIABLE WHOSE PDF IS DEFINED ON A FINITE INTERVAL

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ABSTRACT. Some inequalities for the dispersion of a random variable whose pdf is defined on a finite interval and applications are given.

1. INTRODUCTION

In this note we obtain some inequalities for the dispersion of a continuous random variable X having the probability density function (p.d.f.) f defined on a finite interval $[a, b]$.

Tools used include: Korkine's identity, which plays a central role in the proof of Chebychev's integral inequality for synchronous mappings [24], Hölder's weighted inequality for double integrals and an integral identity connecting the variance $\sigma^2(X)$ and the expectation $E(X)$. Perturbed results are also obtained by using Grüss, Chebyshev and Lupaş inequalities. In Section 4, results from an identity involving a double integral are obtained for a variety of norms.

2. SOME INEQUALITIES FOR DISPERSION

Let $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}_+$ be the p.d.f. of the random variable X and

$$E(X) := \int_a^b t f(t) dt$$

its *expectation* and

$$\begin{aligned} \sigma(X) &= \left[\int_a^b (t - E(X))^2 f(t) dt \right]^{\frac{1}{2}} \\ &= \left[\int_a^b t^2 f(t) dt - [E(X)]^2 \right]^{\frac{1}{2}} \end{aligned}$$

its *dispersion* or *standard deviation*.

The following theorem holds.

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Theorem 1. *With the above assumptions, we have*

$$(2.1) \quad 0 \leq \sigma(X) \leq \begin{cases} \frac{\sqrt{3}(b-a)^2}{6} \|f\|_\infty & \text{provided } f \in L_\infty[a, b]; \\ \frac{\sqrt{2}(b-a)^{1+\frac{1}{q}}}{2[(q+1)(2q+1)]^{\frac{2}{q}}} \|f\|_p & \text{provided } f \in L_p[a, b] \\ & \text{and } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{\sqrt{2}(b-a)}{2}. & \end{cases}$$

Proof. Korkine's identity [24], is

$$(2.2) \quad \begin{aligned} & \int_a^b p(t) dt \int_a^b p(t) g(t) h(t) dt - \int_a^b p(t) g(t) dt \cdot \int_a^b p(t) h(t) dt \\ &= \frac{1}{2} \int_a^b \int_a^b p(t) p(s) (g(t) - g(s)) (h(t) - h(s)) dt ds, \end{aligned}$$

which holds for the measurable mappings $p, g, h : [a, b] \rightarrow \mathbb{R}$ for which the integrals involved in (2.2) exist and are finite. Choose in (2.2) $p(t) = f(t)$, $g(t) = h(t) = t - E(X)$, $t \in [a, b]$ to get

$$(2.3) \quad \sigma^2(X) = \frac{1}{2} \int_a^b \int_a^b f(t) f(s) (t-s)^2 dt ds.$$

It is obvious that

$$(2.4) \quad \begin{aligned} & \int_a^b \int_a^b f(t) f(s) (t-s)^2 dt ds \\ & \leq \sup_{(t,s) \in [a,b]^2} |f(t) f(s)| \int_a^b \int_a^b (t-s)^2 dt ds \\ & = \frac{(b-a)^4}{6} \|f\|_\infty^2 \end{aligned}$$

and then, by (2.3), we obtain the first part of (2.1).

For the second part, we apply Hölder's integral inequality for double integrals to obtain

$$\begin{aligned} & \int_a^b \int_a^b f(t) f(s) (t-s)^2 dt ds \\ & \leq \left(\int_a^b \int_a^b f^p(t) f^p(s) dt ds \right)^{\frac{1}{p}} \left(\int_a^b \int_a^b (t-s)^{2q} dt ds \right)^{\frac{1}{q}} \\ & = \|f\|_p^2 \left[\frac{(b-a)^{2q+2}}{(q+1)(2q+1)} \right]^{\frac{1}{q}}, \end{aligned}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and the second inequality in (2.1) is proved.

For the last part, observe that

$$\begin{aligned} \int_a^b \int_a^b f(t) f(s) (t-s)^2 dt ds & \leq \sup_{(t,s) \in [a,b]^2} |(t-s)|^2 \int_a^b \int_a^b f(t) f(s) dt ds \\ & = (b-a)^2 \end{aligned}$$

as

$$\int_a^b \int_a^b f(t) f(s) dt ds = \int_a^b f(t) dt \int_a^b f(s) ds = 1.$$

■

Using a finer argument, the last inequality in (2.1) can be improved as follows.

Theorem 2. *Under the above assumptions, we have*

$$(2.5) \quad 0 \leq \sigma(X) \leq \frac{1}{2}(b-a).$$

Proof. We use the following Grüss type inequality:

$$(2.6) \quad 0 \leq \frac{\int_a^b p(t) g^2(t) dt}{\int_a^b p(t) dt} - \left(\frac{\int_a^b p(t) g(t) dt}{\int_a^b p(t) dt} \right)^2 \leq \frac{1}{4}(M-m)^2,$$

provided that p, g are measurable on $[a, b]$ and all the integrals in (2.6) exist and are finite, $\int_a^b p(t) dt > 0$ and $m \leq g \leq M$ a.e. on $[a, b]$. For a proof of this inequality see [19].

Choose in (2.6), $p(t) = f(t)$, $g(t) = t - E(X)$, $t \in [a, b]$. Observe that in this case $m = a - E(X)$, $M = b - E(X)$ and then, by (2.6) we deduce (2.5). ■

Remark 1. *The same conclusion can be obtained for the choice $p(t) = f(t)$ and $g(t) = t$, $t \in [a, b]$.*

The following result holds.

Theorem 3. *Let X be a random variable having the p.d.f. given by $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}_+$. Then for any $x \in [a, b]$ we have the inequality:*

$$(2.7) \quad \sigma^2(X) + (x - E(X))^2 \leq \begin{cases} (b-a) \left[\frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2 \right] \|f\|_\infty & \text{provided } f \in L_\infty[a, b]; \\ \left[\frac{(b-x)^{2q+1} + (x-a)^{2q+1}}{2q+1} \right]^{\frac{1}{q}} \|f\|_p & \text{provided } f \in L_p[a, b], p > 1, \\ & \text{and } \frac{1}{p} + \frac{1}{q} = 1; \\ \left(\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right| \right)^2. & \end{cases}$$

Proof. We observe that

$$(2.8) \quad \begin{aligned} \int_a^b (x-t)^2 f(t) dt &= \int_a^b (x^2 - 2xt + t^2) f(t) dt \\ &= x^2 - 2xE(X) + \int_a^b t^2 f(t) dt \end{aligned}$$

and as

$$(2.9) \quad \sigma^2(X) = \int_a^b t^2 f(t) dt - [E(X)]^2,$$

we get, by (2.8) and (2.9),

$$(2.10) \quad [x - E(X)]^2 + \sigma^2(X) = \int_a^b (x-t)^2 f(t) dt,$$

which is of interest in itself too.

We observe that

$$\begin{aligned} \int_a^b (x-t)^2 f(t) dt &\leq \operatorname{ess\,sup}_{t \in [a,b]} |f(t)| \int_a^b (x-t)^2 dt \\ &= \|f\|_\infty \frac{(b-x)^3 + (x-a)^3}{3} \\ &= (b-a) \|f\|_\infty \left[\frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2 \right] \end{aligned}$$

and the first inequality in (2.7) is proved.

For the second inequality, observe that by Hölder's integral inequality,

$$\begin{aligned} \int_a^b (x-t)^2 f(t) dt &\leq \left(\int_a^b f^p(t) dt \right)^{\frac{1}{p}} \left(\int_a^b (x-t)^{2q} dt \right)^{\frac{1}{q}} \\ &= \|f\|_p \left[\frac{(b-x)^{2q+1} + (x-a)^{2q+1}}{2q+1} \right]^{\frac{1}{q}}, \end{aligned}$$

and the second inequality in (2.7) is established.

Finally, observe that,

$$\begin{aligned} \int_a^b (x-t)^2 f(t) dt &\leq \sup_{t \in [a,b]} (x-t)^2 \int_a^b f(t) dt \\ &= \max \left\{ (x-a)^2, (b-x)^2 \right\} \\ &= (\max \{x-a, b-x\})^2 \\ &= \left(\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right)^2, \end{aligned}$$

and the theorem is proved. ■

The following corollaries are easily deduced.

Corollary 1. *With the above assumptions, we have*

$$(2.11) \quad 0 \leq \sigma(X) \leq \begin{cases} (b-a)^{\frac{1}{2}} \left[\frac{(b-a)^2}{12} + \left(E(X) - \frac{a+b}{2}\right)^2 \right]^{\frac{1}{2}} \|f\|_\infty^{\frac{1}{2}} \\ \text{provided } f \in L_\infty[a, b]; \\ \left[\frac{(b-E(X))^{2q+1} + (E(X)-a)^{2q+1}}{2q+1} \right]^{\frac{1}{2q}} \|f\|_p^{\frac{1}{2}} \\ \text{if } f \in L_p[a, b], p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{b-a}{2} + \left| E(X) - \frac{a+b}{2} \right|. \end{cases}$$

Remark 2. *The last inequality in (2.11) is worse than the inequality (2.5), obtained by a technique based on Grüss' inequality.*

The best inequality we can get from (2.7) is that one for which $x = \frac{a+b}{2}$, and this applies for all the bounds as

$$\begin{aligned} \min_{x \in [a,b]} \left[\frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2} \right)^2 \right] &= \frac{(b-a)^2}{12}, \\ \min_{x \in [a,b]} \frac{(b-x)^{2q+1} + (x-a)^{2q+1}}{2q+1} &= \frac{(b-a)^{2q+1}}{2^{2q}(2q+1)}, \end{aligned}$$

and

$$\min_{x \in [a,b]} \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] = \frac{b-a}{2}.$$

Consequently, we can state the following corollary as well.

Corollary 2. *With the above assumptions, we have the inequality:*

$$(2.12) \quad 0 \leq \sigma^2(X) + \left[E(X) - \frac{a+b}{2} \right]^2 \leq \begin{cases} \frac{(b-a)^3}{12} \|f\|_\infty & \text{provided } f \in L_\infty[a, b]; \\ \frac{(b-a)^{2q+1}}{4(2q+1)^{\frac{1}{q}}} \|f\|_p & \text{if } f \in L_p[a, b], p > 1, \\ & \text{and } \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^2}{12}. \end{cases}$$

Remark 3. *from the last inequality in (2.12), we obtain*

$$(2.13) \quad 0 \leq \sigma^2(X) \leq (b - E(X))(E(X) - a) \leq \frac{1}{4}(b-a)^2,$$

which is an improvement on (2.5).

3. PERTURBED RESULTS USING GRÜSS TYPE INEQUALITIES

In 1935, G. Grüss (see for example [26]) proved the following integral inequality which gives an approximation for the integral of a product in terms of the product of the integrals.

Theorem 4. *Let $h, g : [a, b] \rightarrow \mathbb{R}$ be two integrable mappings such that $\phi \leq h(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a, b]$, where $\phi, \Phi, \gamma, \Gamma$ are real numbers. Then,*

$$(3.1) \quad |T(h, g)| \leq \frac{1}{4} (\Phi - \phi) (\Gamma - \gamma),$$

where

$$(3.2) \quad T(h, g) = \frac{1}{b-a} \int_a^b h(x) g(x) dx - \frac{1}{b-a} \int_a^b h(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx$$

and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

For a simple proof of this as well as for extensions, generalisations, discrete variants and other associated material, see [25], and [1]-[21] where further references are given.

A ‘premature’ Grüss inequality is embodied in the following theorem which was proved in [23]. It provides a sharper bound than the above Grüss inequality.

Theorem 5. *Let h, g be integrable functions defined on $[a, b]$ and let $d \leq g(t) \leq D$. Then*

$$(3.3) \quad |T(h, g)| \leq \frac{D-d}{2} |T(h, h)|^{\frac{1}{2}},$$

where $T(h, g)$ is as defined in (3.2).

Theorem 5 will now be used to provide a perturbed rule involving the variance and mean of a p.d.f.

3.1. Perturbed Results Using ‘Premature’ Inequalities. In this subsection we develop some perturbed results.

Theorem 6. *Let X be a random variable having the p.d.f. given by $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}_+$. Then for any $x \in [a, b]$ and $m \leq f(x) \leq M$ we have the inequality*

$$(3.4) \quad |P_V(x)| \quad = \left| \sigma^2(X) + (x - E(X))^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ \leq \frac{M-m}{2} \cdot \frac{(b-a)^2}{\sqrt{45}} \left[\left(\frac{b-a}{2}\right)^2 + 15 \left(x - \frac{a+b}{2}\right) \right]^{\frac{1}{2}} \\ \leq (M-m) \frac{(b-a)^3}{\sqrt{45}}.$$

Proof. Applying the ‘premature’ Grüss result (3.3) by associating $g(t)$ with $f(t)$ and $h(t) = (x-t)^2$, gives, from (3.1)-(3.3)

$$(3.5) \quad \left| \int_a^b (x-t)^2 f(t) dt - \frac{1}{b-a} \int_a^b (x-t)^2 dt \cdot \int_a^b f(t) dt \right| \\ \leq (b-a) \frac{M-m}{2} [T(h, h)]^{\frac{1}{2}},$$

where from (3.2)

$$(3.6) \quad T(h, h) = \frac{1}{b-a} \int_a^b (x-t)^4 dt - \left[\frac{1}{b-a} \int_a^b (x-t)^2 dt \right]^2.$$

Now,

$$(3.7) \quad \frac{1}{b-a} \int_a^b (x-t)^2 dt = \frac{(x-a)^3 + (b-x)^3}{3(b-a)} \\ = \frac{1}{3} \left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2$$

and

$$\frac{1}{b-a} \int_a^b (x-t)^4 dt = \frac{(x-a)^5 + (b-x)^5}{5(b-a)}$$

giving, from (3.6),

$$(3.8) \quad 45T(h, h) = 9 \left[\frac{(x-a)^5 + (b-x)^5}{b-a} \right] - 5 \left[\frac{(x-a)^3 + (b-x)^3}{b-a} \right]^2.$$

Let $A = x - a$ and $B = b - x$ in (3.8) to give

$$\begin{aligned} 45T(h, h) &= 9 \left(\frac{A^5 + B^5}{A + B} \right) - 5 \left(\frac{A^3 + B^3}{A + B} \right)^2 \\ &= 9 [A^4 - A^3B + A^2B^2 - AB^3 + B^4] - 5 [A^2 - AB + B^2]^2 \\ &= (4A^2 - 7AB + 4B^2) (A + B)^2 \\ &= \left[\left(\frac{A + B}{2} \right)^2 + 15 \left(\frac{A - B}{2} \right)^2 \right] (A + B)^2. \end{aligned}$$

Using the facts that $A + B = b - a$ and $A - B = 2x - (a + b)$ gives

$$(3.9) \quad T(h, h) = \frac{(b-a)^2}{45} \left[\left(\frac{b-a}{2} \right)^2 + 15 \left(x - \frac{a+b}{2} \right)^2 \right]$$

and from (3.7)

$$\begin{aligned} \frac{1}{b-a} \int_a^b (x-t)^2 dt &= \frac{A^3 + B^3}{3(A+B)} = \frac{1}{3} [A^2 - AB + B^2] \\ &= \frac{1}{3} \left[\left(\frac{A+B}{2} \right)^2 + 3 \left(\frac{A-B}{2} \right)^2 \right], \end{aligned}$$

giving

$$(3.10) \quad \frac{1}{b-a} \int_a^b (x-t)^2 dt = \frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2} \right)^2.$$

Hence, from (3.5), (3.9) (3.10) and (2.10), the first inequality in (3.4) results. The coarsest uniform bound is obtained by taking x at either end point. Thus the theorem is completely proved. ■

Remark 4. *The best inequality obtainable from (3.4) is at $x = \frac{a+b}{2}$ giving*

$$(3.11) \quad \left| \sigma^2(X) + \left[E(X) - \frac{a+b}{2} \right]^2 - \frac{(b-a)^2}{12} \right| \leq \frac{M-m}{12} \frac{(b-a)^3}{\sqrt{5}}.$$

The result (3.11) is a tighter bound than that obtained in the first inequality of (2.12) since $0 < M - m < 2 \|f\|_\infty$.

For a symmetric p.d.f. $E(X) = \frac{a+b}{2}$ and so the above results would give bounds on the variance.

The following results hold if the p.d.f. $f(x)$ is differentiable, that is, for $f(x)$ absolutely continuous.

Theorem 7. *Let the conditions on Theorem 4 be satisfied. Further, suppose that f is differentiable and is such that*

$$\|f'\|_\infty := \sup_{t \in [a, b]} |f'(t)| < \infty.$$

Then

$$(3.12) \quad |P_V(x)| \leq \frac{b-a}{\sqrt{12}} \|f'\|_\infty \cdot I(x),$$

where $P_V(x)$ is given by the left hand side of (3.4) and,

$$(3.13) \quad I(x) = \frac{(b-a)^2}{\sqrt{45}} \left[\left(\frac{b-a}{2} \right)^2 + 15 \left(x - \frac{a+b}{2} \right)^2 \right]^{\frac{1}{2}}.$$

Proof. Let $h, g : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous and h', g' be bounded. Then Chebychev's inequality holds (see [23])

$$T(h, g) \leq \frac{(b-a)^2}{\sqrt{12}} \sup_{t \in [a, b]} |h'(t)| \cdot \sup_{t \in [a, b]} |g'(t)|.$$

Matić, Pečarić and Ujević [23] using a ‘premature’ Grüss type argument proved that

$$(3.14) \quad T(h, g) \leq \frac{(b-a)}{\sqrt{12}} \sup_{t \in [a, b]} |g'(t)| \sqrt{T(h, h)}.$$

Associating $f(\cdot)$ with $g(\cdot)$ and $(x - \cdot)^2$ with $h(\cdot)$ in (3.13) gives, from (3.5) and (3.9), $I(x) = (b-a) [T(h, h)]^{\frac{1}{2}}$, which simplifies to (3.13) and the theorem is proved. ■

Theorem 8. *Let the conditions of Theorem 6 be satisfied. Further, suppose that f is locally absolutely continuous on (a, b) and let $f' \in L_2(a, b)$. Then*

$$(3.15) \quad |P_V(x)| \leq \frac{b-a}{\pi} \|f'\|_2 \cdot I(x),$$

where $P_V(x)$ is the left hand side of (3.4) and $I(x)$ is as given in (3.13).

Proof. The following result was obtained by Lupas̃ (see [23]). For $h, g : (a, b) \rightarrow \mathbb{R}$ locally absolutely continuous on (a, b) and $h', g' \in L_2(a, b)$, then

$$|T(h, g)| \leq \frac{(b-a)^2}{\pi^2} \|h'\|_2 \|g'\|_2,$$

where

$$\|k\|_2 := \left(\frac{1}{b-a} \int_a^b |k(t)|^2 \right)^{\frac{1}{2}} \quad \text{for } k \in L_2(a, b).$$

Matić, Pečarić and Ujević [23] further show that

$$(3.16) \quad |T(h, g)| \leq \frac{b-a}{\pi} \|g'\|_2 \sqrt{T(h, h)}.$$

Associating $f(\cdot)$ with $g(\cdot)$ and $(x - \cdot)^2$ with h in (3.16) gives (3.15), where $I(x)$ is as found in (3.13), since from (3.5) and (3.9), $I(x) = (b-a) [T(h, h)]^{\frac{1}{2}}$. ■

3.2. Alternate Grüss Type Results for Inequalities Involving the Variance. Let

$$(3.17) \quad S(h(x)) = h(x) - \mathcal{M}(h)$$

where

$$(3.18) \quad \mathcal{M}(h) = \frac{1}{b-a} \int_a^b h(u) du.$$

Then from (3.2),

$$(3.19) \quad \mathcal{T}(h, g) = \mathcal{M}(hg) - \mathcal{M}(h) \mathcal{M}(g).$$

Dragomir and McAndrew [19] have shown, that

$$(3.20) \quad \mathcal{T}(h, g) = \mathcal{T}(S(h), S(g))$$

and proceeded to obtain bounds for a trapezoidal rule. Identity (3.20) is now applied to obtain bounds for the variance.

Theorem 9. *Let X be a random variable having the p.d.f. $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}_+$. Then for any $x \in [a, b]$ the following inequality holds, namely,*

$$(3.21) \quad |P_V(x)| \leq \frac{8}{3} \nu^3(x) \left\| f(\cdot) - \frac{1}{b-a} \right\|_{\infty} \quad \text{if } f \in L_{\infty}[a, b],$$

where $P_V(x)$ is as defined by the left hand side of (3.4), and $\nu = \nu(x) = \frac{1}{3} \left(\frac{b-a}{2} \right)^2 + \left(x - \frac{a+b}{2} \right)^2$.

Proof. Using identity (3.20), associate with $h(\cdot)$, $(x - \cdot)^2$ and $f(\cdot)$ with $g(\cdot)$. Then

$$(3.22) \quad \begin{aligned} & \int_a^b (x-t)^2 f(t) dt - \mathcal{M}\left((x-\cdot)^2\right) \\ &= \int_a^b \left[(x-t)^2 - \mathcal{M}\left((x-\cdot)^2\right) \right] \left[f(t) - \frac{1}{b-a} \right] dt, \end{aligned}$$

where from (3.18),

$$\mathcal{M}\left((x-\cdot)^2\right) = \frac{1}{b-a} \int_a^b (x-t)^2 dt = \frac{1}{3(b-a)} \left[(x-a)^3 + (b-x)^3 \right]$$

and so

$$(3.23) \quad 3\mathcal{M}\left((x-\cdot)^2\right) = \left(\frac{b-a}{2} \right)^2 + 3 \left(x - \frac{a+b}{2} \right)^2.$$

Further, from (3.17),

$$S\left((x-\cdot)^2\right) = (x-t)^2 - \mathcal{M}\left((x-\cdot)^2\right)$$

and so, on using (3.23)

$$(3.24) \quad S\left((x-\cdot)^2\right) = (x-t)^2 - \frac{1}{3} \left(\frac{b-a}{2} \right)^2 - \left(x - \frac{a+b}{2} \right)^2.$$

Now, from (3.22) and using (2.10), (3.23) and (3.24), the following identity is obtained

$$(3.25) \quad \begin{aligned} & \sigma^2(X) + [x - E(X)]^2 - \frac{1}{3} \left[\left(\frac{b-a}{2} \right)^2 + 3 \left(x - \frac{a+b}{2} \right)^2 \right] \\ &= \int_a^b S\left((x-t)^2\right) \left(f(t) - \frac{1}{b-a} \right) dt, \end{aligned}$$

where $S(\cdot)$ is as given by (3.24). Taking the modulus of (3.25) gives

$$(3.26) \quad |P_V(x)| = \left| \int_a^b S\left((x-t)^2\right) \left(f(t) - \frac{1}{b-a} \right) dt \right|.$$

Observe that under different assumptions with regard to the norms of the p.d.f. $f(x)$ we may obtain a variety of bounds.

For $f \in L_\infty [a, b]$ then

$$(3.27) \quad |P_V(x)| \leq \left\| f(\cdot) - \frac{1}{b-a} \right\|_\infty \int_a^b |S((x-t)^2)| dt.$$

Now, let

$$(3.28) \quad S((x-t)^2) = (t-x)^2 - \nu^2 = (t-X_-)(t-X_+),$$

where

$$(3.29) \quad \begin{aligned} \nu^2 &= \mathcal{M}((x-\cdot)^2) = \frac{(x-a)^3 + (b-x)^3}{3(b-a)} \\ &= \frac{1}{3} \left(\frac{b-a}{2} \right)^2 + \left(x - \frac{a+b}{2} \right)^2, \end{aligned}$$

and

$$(3.30) \quad X_- = x - \nu, \quad X_+ = x + \nu.$$

Then,

$$(3.31) \quad \begin{aligned} H(t) &= \int S((x-t)^2) dt = \int [(t-x)^2 - \nu^2] dt \\ &= \frac{(t-x)^3}{3} - \nu^2 t + k \end{aligned}$$

and so from (3.31) and using (3.28) - (3.29) gives,

$$(3.32) \quad \begin{aligned} &\int_a^b |S((x-t)^2)| dt \\ &= H(X_-) - H(a) - [H(X_+) - H(X_-)] + [H(b) - H(X_+)] \\ &= 2[H(X_-) - H(X_+)] + H(b) - H(a) \\ &= 2 \left\{ -\frac{\nu^3}{3} - \nu^2 X_- - \frac{\nu^3}{3} + \nu^2 X_+ \right\} + \frac{(b-x)^3}{3} - \nu^2 b + \frac{(x-a)^3}{3} + \nu^2 a \\ &= 2 \left[2\nu^3 - \frac{2}{3}\nu^3 \right] + \frac{(b-x)^3 + (x-a)^3}{3} - \nu^2(b-a) \\ &= \frac{8}{3}\nu^3. \end{aligned}$$

Thus, substituting into (3.27), (3.26) and using (3.29) readily produces the result (3.21) and the theorem is proved. ■

Remark 5. Other bounds may be obtained for $f \in L_p [a, b]$, $p \geq 1$ however obtaining explicit expressions for these bounds is somewhat intricate and will not be considered further here. They involve the calculation of

$$\sup_{t \in [a, b]} |(t-x)^2 - \nu^2| = \max \left\{ |(x-a)^2 - \nu^2|, \nu^2, |(b-x)^2 - \nu^2| \right\}$$

for $f \in L_1 [a, b]$ and

$$\left(\int_a^b |(t-x)^2 - \nu^2|^q dt \right)^{\frac{1}{q}}$$

for $f \in L_p [a, b]$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, where ν^2 is given by (3.29).

4. SOME INEQUALITIES FOR ABSOLUTELY CONTINUOUS P.D.F.s

We start with the following lemma which is interesting in itself.

Lemma 1. *Let X be a random variable whose probability density function $f : [a, b] \rightarrow \mathbb{R}_+$ is absolutely continuous on $[a, b]$. Then we have the identity*

$$(4.1) \quad \begin{aligned} & \sigma^2(X) + [E(X) - x]^2 \\ &= \frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2 + \frac{1}{b-a} \int_a^b \int_a^b (t-x)^2 p(t,s) f'(s) ds dt, \end{aligned}$$

where the kernel $p : [a, b]^2 \rightarrow \mathbb{R}$ is given by

$$p(t, s) := \begin{cases} s - a & \text{if } a \leq s \leq t \leq b \\ s - b & \text{if } a \leq t < s \leq b \end{cases},$$

for all $x \in [a, b]$.

Proof. We use the identity (see (2.10))

$$(4.2) \quad \sigma^2(X) + [E(X) - x]^2 = \int_a^b (x-t)^2 f(t) dt$$

for all $x \in [a, b]$.

On the other hand, we know that (see for example [22] for a simple proof using integration by parts)

$$(4.3) \quad f(t) = \frac{1}{b-a} \int_a^b f(s) ds + \frac{1}{b-a} \int_a^b p(t,s) f'(s) ds$$

for all $t \in [a, b]$.

Substituting (4.3) in (4.2) we obtain

$$(4.4) \quad \begin{aligned} & \sigma^2(X) + [E(X) - x]^2 \\ &= \int_a^b (t-x)^2 \left[\frac{1}{b-a} \int_a^b f(s) ds + \frac{1}{b-a} \int_a^b p(t,s) f'(s) ds \right] dt \\ &= \frac{1}{b-a} \cdot \frac{1}{3} [(x-a)^3 + (b-x)^3] + \frac{1}{b-a} \int_a^b \int_a^b (t-x)^2 p(t,s) f'(s) ds dt. \end{aligned}$$

Taking into account the fact that

$$\frac{1}{3} [(x-a)^3 + (b-x)^3] = \frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2, \quad x \in [a, b],$$

then, by (4.4) we deduce the desired result (4.1). ■

The following inequality for P.D.F.s which are absolutely continuous and have the derivatives essentially bounded holds.

Theorem 10. *If $f : [a, b] \rightarrow \mathbb{R}_+$ is absolutely continuous on $[a, b]$ and $f' \in L_\infty [a, b]$, i.e., $\|f'\|_\infty := \text{ess sup}_{t \in [a, b]} |f'(t)| < \infty$, then we have the inequality:*

$$(4.5) \quad \left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ \leq \frac{(b-a)^2}{3} \left[\frac{(b-a)^2}{10} + \left(x - \frac{a+b}{2}\right)^2 \right] \|f'\|_\infty$$

for all $x \in [a, b]$.

Proof. Using Lemma 1, we have

$$\left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ = \frac{1}{b-a} \left| \int_a^b \int_a^b (t-x)^2 p(t,s) f'(s) ds dt \right| \\ \leq \frac{1}{b-a} \int_a^b \int_a^b (t-x)^2 |p(t,s)| |f'(s)| ds dt \\ \leq \frac{\|f'\|_\infty}{b-a} \int_a^b \int_a^b (t-x)^2 |p(t,s)| ds dt.$$

We have

$$I \quad : \quad = \int_a^b \int_a^b (t-x)^2 |p(t,s)| ds dt \\ = \int_a^b (t-x)^2 \left[\int_a^t (s-a) ds + \int_t^b (b-s) ds \right] dt \\ = \int_a^b (t-x)^2 \left[\frac{(t-a)^2}{2} + \frac{(b-t)^2}{2} \right] dt \\ = \frac{1}{2} \left[\int_a^b (t-x)^2 (t-a)^2 dt + \int_a^b (t-x)^2 (b-t)^2 dt \right] \\ = \frac{(I_a + I_b)}{2}.$$

Let $A = x - a$, $B = b - x$ then

$$I_a \quad = \quad \int_a^b (t-x)^2 (t-a)^2 dt \\ = \int_0^{b-a} (u^2 - 2Au + A^2) u^2 du \\ = \frac{(b-a)^3}{3} \left[A^2 - \frac{3}{2}A(b-a) + \frac{3}{5}(b-a)^2 \right]$$

and

$$\begin{aligned}
I_b &= \int_a^b (t-x)^2 (b-t)^2 dt \\
&= \int_0^{b-a} (u^2 - 2Bu + B^2) u^2 du \\
&= \frac{(b-a)^3}{3} \left[B^2 - \frac{3}{2}B(b-a) + \frac{3}{5}(b-a)^2 \right]
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{I_a + I_b}{2} &= \frac{(b-a)^3}{3} \left[\frac{A^2 + B^2}{2} - \frac{3}{4}(A+B)(b-a) + \frac{3}{5}(b-a)^2 \right] \\
&= \frac{(b-a)^3}{3} \left[\left(\frac{b-a}{2} \right)^2 + \left(x - \frac{a+b}{2} \right)^2 - 3 \frac{(b-a)^2}{20} \right] \\
&= \frac{(b-a)^3}{3} \left[\frac{(b-a)^2}{10} + \left(x - \frac{a+b}{2} \right)^2 \right]
\end{aligned}$$

and the theorem is proved. ■

The best inequality we can get from (4.5) is embodied in the following corollary.

Corollary 3. *If f is as in Theorem 10, then we have*

$$(4.6) \quad \left| \sigma^2(X) + \left[E(X) - \frac{a+b}{2} \right]^2 - \frac{(b-a)^2}{12} \right| \leq \frac{(b-a)^4}{30} \|f'\|_\infty.$$

We now analyze the case where f' is a Lebesgue p -integrable mapping with $p \in (1, \infty)$.

Remark 6. *The results of Theorem 10 may be compared with those of Theorem 7. It may be shown that both bounds are convex and symmetric about $x = \frac{a+b}{2}$. Further, the bound given by the 'premature' Chebychev approach, namely from (3.12)-(3.13) is tighter than that obtained by the current approach (4.5) which may be shown from the following. Let these bounds be described by B_p and B_c so that, neglecting the common terms*

$$B_p = \frac{b-a}{2\sqrt{15}} \left[\left(\frac{b-a}{2} \right)^2 + 15Y \right]^{\frac{1}{2}}$$

and

$$B_c = \frac{(b-a)^2}{100} + Y,$$

where

$$Y = \left(x - \frac{a+b}{2} \right)^2.$$

It may be shown through some straightforward algebra that $B_c^2 - B_p^2 > 0$ for all $x \in [a, b]$ so that $B_c > B_p$.

The current development does however have the advantage that the identity (4.1) is satisfied, thus allowing bounds for $L_p[a, b]$, $p \geq 1$ rather than the infinity norm.

Theorem 11. *If $f : [a, b] \rightarrow \mathbb{R}_+$ is absolutely continuous on $[a, b]$ and $f' \in L_p$, i.e.,*

$$\|f'\|_p := \left(\int_a^b |f'(t)|^p dt \right)^{\frac{1}{p}} < \infty, \quad p \in (1, \infty)$$

then we have the inequality

$$(4.7) \quad \left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ \leq \frac{\|f'\|_p}{(b-a)^{\frac{1}{p}} (q+1)^{\frac{1}{q}}} \left[(x-a)^{3q+2} \tilde{B}\left(\frac{b-a}{x-a}, 2q+1, q+2\right) \right. \\ \left. + (b-x)^{3q+2} \tilde{B}\left(\frac{b-a}{b-x}, 2q+1, q+2\right) \right]$$

for all $x \in [a, b]$, when $\frac{1}{p} + \frac{1}{q} = 1$ and $\tilde{B}(\cdot, \cdot, \cdot)$ is the quasi incomplete Euler's Beta mapping:

$$\tilde{B}(z; \alpha, \beta) := \int_0^z (u-1)^{\alpha-1} u^{\beta-1} du, \quad \alpha, \beta > 0, \quad z \geq 1.$$

Proof. Using Lemma 1, we have, as in Theorem 10, that

$$(4.8) \quad \left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ \leq \frac{1}{b-a} \int_a^b \int_a^b (t-x)^2 |p(t, s)| |f'(s)| ds dt.$$

Using Hölder's integral inequality for double integrals, we have

$$(4.9) \quad \int_a^b \int_a^b (t-x)^2 |p(t, s)| |f'(s)| ds dt \\ \leq \left(\int_a^b \int_a^b |f'(s)|^p ds dt \right)^{\frac{1}{p}} \left(\int_a^b \int_a^b (t-x)^{2q} |p(t, s)|^q ds dt \right)^{\frac{1}{q}} \\ = (b-a)^{\frac{1}{p}} \|f'\|_p \left(\int_a^b \int_a^b (t-x)^{2q} |p(t, s)|^q ds dt \right)^{\frac{1}{q}},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

We have to compute the integral

$$(4.10) \quad D : = \int_a^b \int_a^b (t-x)^{2q} |p(t, s)|^q ds dt \\ = \int_a^b (t-x)^{2q} \left[\int_a^t (s-a)^q ds + \int_t^b (b-s)^q ds \right] dt \\ = \int_a^b (t-x)^{2q} \left[\frac{(t-a)^{q+1} + (b-t)^{q+1}}{q+1} \right] dt \\ = \frac{1}{q+1} \left[\int_a^b (t-x)^{2q} (t-a)^{q+1} dt + \int_a^b (t-x)^{2q} (b-t)^{q+1} dt \right].$$

Define

$$(4.11) \quad E := \int_a^b (t-x)^{2q} (t-a)^{q+1} dt.$$

If we consider the change of variable $t = (1-u)a + ux$, we have $t = a$ implies $u = 0$ and $t = b$ implies $u = \frac{b-a}{x-a}$, $dt = (x-a) du$ and then

$$(4.12) \quad \begin{aligned} E &= \int_0^{\frac{b-a}{x-a}} [(1-u)a + ux - x]^{2q} [(1-u)a + ux - a] (x-a) du \\ &= (x-a)^{3q+2} \int_0^{\frac{b-a}{x-a}} (u-1)^{2q} u^{q+1} du \\ &= (x-a)^{3q+2} \tilde{B}\left(\frac{b-a}{x-a}, 2q+1, q+2\right). \end{aligned}$$

Define

$$(4.13) \quad F := \int_a^b (t-x)^{2q} (b-t)^{q+1} dt.$$

If we consider the change of variable $t = (1-v)b + vx$, we have $t = b$ implies $v = 0$, and $t = a$ implies $v = \frac{b-a}{b-x}$, $dt = (x-b) dv$ and then

$$(4.14) \quad \begin{aligned} F &= \int_{\frac{b-a}{b-x}}^0 [(1-v)b + vx - x]^{2q} [b - (1-v)b - vx]^{q+1} (x-b) dv \\ &= (b-x)^{3q+2} \int_0^{\frac{b-a}{b-x}} (v-1)^{2q} v^{q+1} dv \\ &= (b-x)^{3q+2} \tilde{B}\left(\frac{b-a}{b-x}, 2q+1, q+2\right). \end{aligned}$$

Now, using the inequalities (4.8)-(4.9) and the relations (4.10)-(4.14), since $D = \frac{1}{q+1} (E + F)$, we deduce the desired estimate (4.7). ■

The following corollary is natural to be considered.

Corollary 4. *Let f be as in Theorem 11. Then, we have the inequality:*

$$(4.15) \quad \begin{aligned} &\left| \sigma^2(X) + \left[E(X) - \frac{a+b}{2} \right]^2 - \frac{(b-a)^2}{12} \right| \\ &\leq \frac{\|f'\|_p (b-a)^{2+\frac{3}{q}}}{(q+1)^{\frac{1}{q}} 2^{3+\frac{2}{q}}} [B(2q+1, q+1) + \Psi(2q+1, q+2)]^{\frac{1}{q}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$ and $B(\cdot, \cdot)$ is Euler's Beta mapping and $\Psi(\alpha, \beta) := \int_0^1 u^{\alpha-1} (u+1)^{\beta-1} du$, $\alpha, \beta > 0$.

Proof. In (4.7) put $x = \frac{a+b}{2}$.

The left side is clear.

Now

$$\begin{aligned}\tilde{B}(2, 2q + 1, q + 2) &= \int_0^2 (u - 1)^{2q} u^{q+1} du \\ &= \int_0^1 (u - 1)^{2q} u^{q+1} du + \int_1^2 (u - 1)^{2q} u^{q+1} du \\ &= B(2q + 1, q + 2) + \Psi(2q + 1, q + 2).\end{aligned}$$

The right hand side of (4.7) is thus:-

$$\begin{aligned}& \frac{\|f'\|_p \left(\frac{b-a}{2}\right)^{\frac{3q+2}{q}}}{(b-a)^{\frac{1}{p}} (q+1)^{\frac{1}{q}}} [2B(2q+1, q+2) + 2\Psi(2q+1, q+2)]^{\frac{1}{q}} \\ &= \frac{\|f'\|_p (b-a)^{2+\frac{3}{q}}}{(q+1)^{\frac{1}{q}} 2^{3+\frac{2}{q}}} [B(2q+1, q+2) + \Psi(2q+1, q+2)]^{\frac{1}{q}}\end{aligned}$$

and the corollary is proved. ■

Finally, as f is absolutely continuous, $f' \in L_1[a, b]$ and $\|f'\|_1 = \int_a^b |f'(t)| dt$, and we can state the following theorem.

Theorem 12. *If the p.d.f., $f : [a, b] \rightarrow \mathbb{R}_+$ is absolutely continuous on $[a, b]$, then*

$$(4.16) \quad \begin{aligned}& \left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ & \leq \|f'\|_1 (b-a) \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right| \right]^2\end{aligned}$$

for all $x \in [a, b]$.

Proof. As above, we can state that

$$\begin{aligned}& \left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ & \leq \frac{1}{b-a} \int_a^b \int_a^b (t-x)^2 |p(t, s)| |f'(s)| ds dt \\ & \leq \sup_{(t,s) \in [a,b]^2} [(t-x)^2 |p(t, s)|] \frac{1}{b-a} \int_a^b \int_a^b |f'(s)| ds dt \\ & = \|f'\|_1 G\end{aligned}$$

where

$$\begin{aligned}G &: = \sup_{(t,s) \in [a,b]^2} [(t-x)^2 |p(t, s)|] \leq (b-a) \sup_{t \in [a,b]} (t-x)^2 \\ &= (b-a) [\max(x-a, b-x)]^2 \\ &= (b-a) \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right| \right]^2,\end{aligned}$$

and the theorem is proved. ■

It is clear that the best inequality we can get from (4.16) is the one when $x = \frac{a+b}{2}$, giving the following corollary.

Corollary 5. *With the assumptions of Theorem 12, we have:*

$$(4.17) \quad \left| \sigma^2(X) + \left[E(X) - \frac{a+b}{2} \right]^2 - \frac{(b-a)^2}{12} \right| \leq \frac{(b-a)^3}{4} \|f'\|_1.$$

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