MONOTONICITY RESULTS AND INEQUALITIES FOR THE GAMMA AND INCOMPLETE GAMMA FUNCTIONS

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Abstract. In the article, using the monotonicity and inequalities of the generalized weighted mean values with two parameters, we prove that the functions $\big[\Gamma(s)/\Gamma(r)\big]^{1/(s-r)}$, $\big[\Gamma(s,x)/\Gamma(r,x)\big]^{1/(s-r)}$ and $\big[\gamma(s,x)/\gamma(r,x)\big]^{1/(s-r)}$ are increasing in $r>0$, $s>0$ and $x>0$, where $\Gamma(s)$, $\Gamma(s,x)$ and $\gamma(s,x)$ denote the gamma and incomplete gamma functions with usual notation. From this, some monotonicity results and inequalities for the gamma or incomplete gamma functions are deduced or extended, a unified proof of some known results for the gamma function is given.

1. Introduction

It is known that the incomplete gamma function is defined and denoted for $\operatorname{Re} z>0$ by

$$\Gamma(z,x)=\int_x^\infty t^{z-1}\mathrm{e}^{-t}\mathrm{d}t,\quad \gamma(z,x)=\int_0^x t^{z-1}\mathrm{e}^{-t}\mathrm{d}t, \tag{1}$$

and $\Gamma(z,0)=\Gamma(z)$ is called the gamma function, $\Gamma(0,x)=E_1(x)$ the exponential integral.

Some monotonicity results and inequalities for the function $\Gamma(x+\lambda)/\Gamma(x+1)$ and the gamma function $\Gamma(x)$ with usual notation, where $x>0$ and $0<\lambda<1$ is independent of x, have been studied by many authors, cf. [1]–[11], [14]–[16], [21], [23] and [28].

Recently the author in [18] established the generalized weighted mean values $M_{p,f}(r,s;x,y)$ of a positive function f with two parameters $r,s\in\mathbb{R}$ and nonnegative weight $p\not\equiv 0$ for $x,y\in\mathbb{R}$ by

$$M_{p,f}(r,s;x,y)=\left(\frac{\int_x^y p(u)f^s(u)\mathrm{d}u}{\int_x^y p(u)f^r(u)\mathrm{d}u}\right)^{1/(s-r)},\qquad (r-s)(x-y)\neq 0; \tag{2}$$

$$M_{p,f}(r,r;x,y)=\exp\left(\frac{\int_x^y p(u)f^r(u)\ln f(u)\mathrm{d}u}{\int_x^y p(u)f^r(u)\mathrm{d}u}\right),\qquad x-y\neq 0; \tag{3}$$

$$M_{p,f}(r,s;x,x)=f(x),\qquad x=y.$$

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For convenience, we write, shifting notation to suit the context,

$$M_{p,f}(r,s;x,y) = M_{p,f}(r,s) = M_{p,f}(x,y) = M_{p,f}. \tag{4}$$

Set $p(u) \equiv 1$, $f(u) = u$ and $x, y > 0$, then the generalized weighted mean values are reduced to the extended mean values $E(r,s;x,y)$ defined as

$$E(r,s;x,y) = \left[\frac{r}{s}\cdot\frac{y^s - x^s}{y^r - x^r}\right]^{1/(s-r)}, \qquad rs(r-s)(x-y) \neq 0; \tag{5}$$

$$E(r,0;x,y) = \left[\frac{1}{r}\cdot\frac{y^r - x^r}{\ln y - \ln x}\right]^{1/r}, \qquad r(x-y) \neq 0; \tag{6}$$

$$E(r,r;x,y) = \mathrm{e}^{-1/r}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r - y^r)}, \qquad r(x-y) \neq 0; \tag{7}$$

$$E(0,0;x,y) = \sqrt{xy}, \qquad x \neq y;$$

$$E(r,s;x,x) = x, \qquad x = y.$$

Remark 1. Many proofs of monotonicities for $E(r,s;x,y)$ and $M_{p,f}(r,s;x,y)$ have been presented by some authors, for details, please refer to [12, 18, 19, 24, 26, 29]. The logarithmic convexity of $E(r,s;x,y)$ is investigated in [20].

In this article, using the monotonicity and inequalities of the generalized weighted mean values $M_{p,f}(r,s;x,y)$, we verified that the functions $\big[\Gamma(s)/\Gamma(r)\big]^{1/(s-r)}$, $\big[\Gamma(s,x)/\Gamma(r,x)\big]^{1/(s-r)}$ and $\big[\gamma(s,x)/\gamma(r,x)\big]^{1/(s-r)}$ are increasing in $r > 0$, $s > 0$ and $x > 0$, respectively. In consequence, some monotonicity results and inequalities for the gamma or incomplete gamma functions are deduced or extended, a unified proof of some well-known results for the gamma function is provided.

2. Monotonicity Results and Inequalities

Lemma 1 ([22])**.** *Suppose $f(t)$ is a positive differentiable function and $p(t) \not\equiv 0$ an integrable nonnegative weight on the interval $[a,b]$, if $f'(t)$ and $f'(t)/p(t)$ are integrable and both increasing or both decreasing, then for all real numbers r and s, we have*

$$M_{p,f}(r,s;a,b) < E\big(r+1, s+1; f(a), f(b)\big); \tag{8}$$

if one of the functions $f'(t)$ or $f'(t)/p(t)$ is nondecreasing and the other nonincreasing, then the inequality (8) *reverses.*

Theorem 1. *For any given $x > 0$, the function $s\gamma(s,x)/x^s$ is decreasing in $s > 0$.*

Proof. Set $p(t) = \mathrm{e}^{-t}$, $f(t) = t$, $t \in (0,x)$ in Lemma 1, then, for $s > r > 0$, we get

$$\left(\frac{\int_0^x t^{s-1}\mathrm{e}^{-t}\mathrm{d}t}{\int_0^x t^{r-1}\mathrm{e}^{-t}\mathrm{d}t}\right)^{1/(s-r)} \leq \left(\frac{r}{s}\cdot\frac{x^s}{x^r}\right)^{1/(s-r)}.$$

Simplifying above inequality yields

$$\frac{s\gamma(s,x)}{x^s} \leq \frac{r\gamma(r,x)}{x^r}.$$

This implies Theorem 1. □

Lemma 2 ([18, 27])**.** *Let $p(u) \not\equiv 0$ be a nonnegative and continuous function, $f(u)$ a positive and continuous function. Then $M_{p,f}(r,s)$ increases with both r and s.*

Theorem 2. *The function* $\big[\Gamma(s)/\Gamma(r)\big]^{1/(s-r)}$ *is increasing with* $r>0$ *and* $s>0$.

Proof. This follows from Lemma 2 applied to $p(u)=\mathrm{e}^{-u}$, $f(u)=u$, $u\in(0,+\infty)$ and standard arguments. $\square$

Corollary 1. *The functions* $\big[\Gamma(r)\big]^{1/(r-1)}$ *and* $\psi(r)=\Gamma'(r)/\Gamma(r)$, *the logarithmic derivative of the gamma function* $\Gamma(r)$, *are increasing in* $r>0$. *Hence* $\Gamma(r)$ *is a logarithmically convex function in the interval* $(0,+\infty)$.

Remark 2. In [8, 14], among other things, the following monotonicity results were obtained

$$\begin{gathered}\big[\Gamma(1+k)\big]^{1/k}<\big[\Gamma(2+k)\big]^{1/(k+1)},\quad k\in\mathbb{N};\\ \Big[\Gamma\Big(1+\frac{1}{x}\Big)\Big]^{x}\text{ decreases with } x>0.\end{gathered}$$

Clearly, our Theorem 2 and Corollary 1 generalize and extend these results for the range of the argument.

Corollary 2. *The following inequalities hold for* $s>r>0$

$$\exp\big[(s-r)\psi(s)\big]>\frac{\Gamma(s)}{\Gamma(r)}>\exp\big[(s-r)\psi(r)\big],\tag{9}$$

$$\mathrm{e}^{cr}<\Gamma(r+1)<\exp\big[r\psi(r+1)\big],\tag{10}$$

where $c=0.5772\cdots$ *is the Euler's constant.*

Proof. These follow from standard arguments and the following relationships

$$\begin{gathered}M_{p,f}(s,s)>M_{p,f}(r,s)>M_{p,f}(r,r),\\ M_{p,f}(r,r)>M_{p,f}(r,0)>M_{p,f}(0,0)\end{gathered}$$

for $s>r>0$. $\square$

Remark 3. The ratio $\Gamma(s)/\Gamma(r)$ has been researched by many mathematicians. W. Gautschi showed in [5] for $0<s<1$ and $n\in\mathbb{N}$ that

$$n^{1-s}<\frac{\Gamma(n+1)}{\Gamma(n+s)}<\exp\big[(1-s)\psi(n+1)\big].\tag{11}$$

A strenghened upper bound was given by T. Erber in [4]

$$\frac{\Gamma(n+1)}{\Gamma(n+s)}<\frac{4(n+s)(n+1)^{1-s}}{4n+(s+1)^2},\quad 0<s<1,\quad n\in\mathbb{N}.\tag{12}$$

J. D. Kečkić and P. M. Vasić gave in [6] the inequalities below

$$\frac{b^{b-1}}{a^{a-1}}\cdot\mathrm{e}^{a-b}<\frac{\Gamma(b)}{\Gamma(a)}<\frac{b^{b-1/2}}{a^{a-1/2}}\cdot\mathrm{e}^{a-b},\quad 0<a<b.\tag{13}$$

The following closer bounds were proved for $0<s<1$ and $x\geq1$ by D. Kershaw in [7]

$$\exp\big[(1-s)\psi(x+s^{1/2})\big]<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\exp\Big[(1-s)\psi\Big(x+\frac{s+1}{2}\Big)\Big],\tag{14}$$

$$\Big(x+\frac{s}{2}\Big)^{1-s}<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\Big[x-\frac{1}{2}+\Big(s-\frac{1}{4}\Big)^{1/2}\Big]^{1-s}.\tag{15}$$

Inequalities for the incomplete gamma function are given for $a > 0$ in [16, p. 526] and [28, pp. 442–443] as follows

$$a\gamma(a,a) > \gamma(a+1,a+1), \tag{16}$$

$$\frac{\gamma(a,a)}{\Gamma(a)} > \frac{\gamma(a+1,a+1)}{\Gamma(a+1)}, \tag{17}$$

$$\frac{\gamma(a,a)}{\Gamma(a)} > \frac{1}{2}. \tag{18}$$

More inequalities and some monotonicity results for the same quotient could be found in [9, 10, 11, 15]. Similar results can be found in [13].

It is easy to see that inequalities in (9) of Corollary 2 extend the range of arguments of above inequalities (11)–(15) but (13).

To the best of my knowledge, inequalities in (10) are new. In [1, 2, 3], Horst Alzer established many inequalities for the gamma function. In [21, 23] the authors found some inequalities of the incomplete gamma function by the Tchebycheff's integral inequality and Hermite-Hadamard's inequality.

Lemma 3 ([18])**.** *Let $p(u) \not\equiv 0$ be a nonnegative and continuous function, $f(u)$ a positive, increasing* (*or decreasing, respectively*) *and continuous function. Then $M_{p,f}(x,y)$ increases* (*or decreases, respectively*) *with respect to either x or y.*

Theorem 3. *For $s > r > 0$ and $x > 0$, $\big[\gamma(s,x)/\gamma(r,x)\big]^{1/(s-r)}$ and $\big[\Gamma(s,x)/\Gamma(r,x)\big]^{1/(s-r)}$ increase with either x or r and s. Therefore, $\gamma(s,x)/x^{s-1}$ decreases and $\Gamma(s,x)/x^{s-1}$ increases with $s > 0$, respectively.*

Proof. The first part is a simple consequence of Lemma 2 and 3. The second part is concluded from differentiating $\gamma(s,x)/\gamma(r,x)$ and $\Gamma(s,x)/\Gamma(r,x)$ with respect to x and standard argument. $\square$

Corollary 3. *The incomplete gamma functions $\gamma(r,x)$ and $\Gamma(r,x)$ are logarithmically convex with respect to $r > 0$ for $x > 0$. The function $\big[\Gamma(r,x)/E_1(x)\big]^{1/r}$ is increasing in $r > 0$ and $x > 0$, where $E_1(x)$ denotes the exponential integral. Therefore, the functions $\Gamma(s+\theta)/\Gamma(r+\theta)$, $\Gamma(s+\theta,x)/\Gamma(r+\theta,x)$ and $\gamma(s+\theta,x)/\gamma(r+\theta,x)$ are increasing with θ for $s > r > 0$ and $x > 0$.*

Lemma 4 ([18])**.** *Let $p_1(u) \not\equiv 0$ and $p_2(u) \not\equiv 0$ be nonnegative and integrable functions on the interval between x and y, $f(u)$ a positive and integrable function, the ratio $p_1(u)/p_2(u)$ an integrable function, $p_1(u)/p_2(u)$ and $f(u)$ both increasing or both decreasing. Then*

$$M_{p_1,f}(r,s;x,y) \geq M_{p_2,f}(r,s;x,y) \tag{19}$$

If one of the functions of $f(u)$ or $p_1(u)/p_2(u)$ is nonincreasing and the other nondecreasing, then inequality (19) *is reversed.*

Theorem 4. *Let $g(t)$ be an integrable positive function such that $\mathrm{e}^t g(t)$ decreasing, then*

$$\frac{\gamma(s,x)}{\gamma(r,x)} \ge \frac{\int_0^x t^{s-1} g(t)\mathrm{d}t}{\int_0^x t^{r-1} g(t)\mathrm{d}t}, \tag{20}$$

$$\frac{\Gamma(s,x)}{\Gamma(r,x)} \ge \frac{\int_x^\infty t^{s-1} g(t)\mathrm{d}t}{\int_x^\infty t^{r-1} g(t)\mathrm{d}t}, \tag{21}$$

$$\frac{\Gamma(s)}{\Gamma(r)} \ge \frac{\int_0^\infty t^{s-1} g(t)\mathrm{d}t}{\int_0^\infty t^{r-1} g(t)\mathrm{d}t} \tag{22}$$

hold for $s > r > 0$ and $x > 0$. If $\mathrm{e}^t g(t)$ is increasing, then the above inequalities reverse.

Proof. These are special cases of inequality (19) in Lemma 4 applied to $f(t) = t$, $p_1(t) = \mathrm{e}^{-t}$ and $p_2(t) = g(t)$. □

Lemma 5 ([18])**.** *Let $p(u) \not\equiv 0$ be a nonnegative and integrable function, and $f_1(u)$ and $f_2(u)$ positive and integrable functions on the interval between x and y. If the ratio $f_1(u)/f_2(u)$ and $f_2(u)$ are integrable and both increasing or both decreasing, then*

$$M_{p,f_1}(r,s;x,y) \ge M_{p,f_2}(r,s;x,y) \tag{23}$$

holds for $r,s \ge 0$ or $r \ge 0 \ge s$, and $f_1(u)/f_2(u) \ge 1$. The inequality (23) *is reversed for $r,s \le 0$ or $s \ge 0 \ge r$, and $f_1(u)/f_2(u) \le 1$.*

If one of the functions of $f_2(u)$ or $f_1(u)/f_2(u)$ is nonincreasing and the other nondecreasing, then inequality (23) *is valid for $r,s \ge 0$ or $s \ge 0 \ge r$, and $f_1(u)/f_2(u) \ge 1$; the inequality* (23) *reverses for $r,s \ge 0$ or $r \ge 0 \ge s$, and $f_1(u)/f_2(u) \le 1$.*

Theorem 5. *Let $f(u)$ be a positive and integrable function on $(0,+\infty)$. If $f(u)/u > 1$ is increasing, then, for $s > r > 0$ and $x > 0$, we have*

$$\frac{\Gamma(s+1)}{\Gamma(r+1)} \le \frac{\int_0^\infty f^s(u)\mathrm{e}^{-u}\mathrm{d}u}{\int_0^\infty f^r(u)\mathrm{e}^{-u}\mathrm{d}u}, \tag{24}$$

$$\frac{\Gamma(s+1,x)}{\Gamma(r+1,x)} \le \frac{\int_x^\infty f^s(u)\mathrm{e}^{-u}\mathrm{d}u}{\int_x^\infty f^r(u)\mathrm{e}^{-u}\mathrm{d}u}, \tag{25}$$

$$\frac{\gamma(s+1,x)}{\gamma(r+1,x)} \le \frac{\int_0^x f^s(u)\mathrm{e}^{-u}\mathrm{d}u}{\int_0^x f^r(u)\mathrm{e}^{-u}\mathrm{d}u}. \tag{26}$$

If $f(u)/u < 1$ is decreasing, the above inequalities reverse for $s > r > 0$ and $x > 0$.

Proof. This is a direct consequence of Lemma 5 applied to $f_1(u) = f(u)$, $f_2(u) = u$ and $p(u) = \mathrm{e}^{-u}$. □

Remark 4. Recently, using the approach by A. Laforgia and S. Sismondi in [11], some more general inequalities of the functions $\int_0^x \mathrm{e}^{p^t}\mathrm{d}t$ and $\int_0^x \mathrm{e}^{-p^t}\mathrm{d}t$ for $p > 0$ and $x > 0$ are obtained in [25].

References

1. H. Alzer, *On some inequalities for the incomplete gamma function*, Math. Comp. **66** (1997), 771–778.
2. H. Alzer, *On some inequalities for the gamma and psi functions*, Math. Comp. **66** (1997), 373–389.
3. H. Alzer, *On some inequalities involving* $(n!)^{1/n}$, Rocky Mountain Journal of Mathematics **24** (1994), no. 3, 867–873.
4. T. Erber, *The gamma function inequalities of Gurland and Gautschi*, Skand. Aktuarietidskr. **1960** (1961), 27–28.
5. W. Gautschi, *Some elementary inequalities relating to the gamma and incomplete gamma function*, J. Math. Phys. **38** (1959), 77–81.
6. J. D. Kečlić and P. M. Vasić, *Some inequalities for the gamma function*, Publ. Inst. Math. Beograd N. S. **11** (1971), 107–114.
7. D. Kershaw, *Some extensions of W. Gautschi's inequalities for the gamma function*, Math. Comp. **41** (1983), 607–611.
8. D. Kershaw and A. Laforgia, *Monotonicity results for the gamma function*, Atti della Accademia delle Scienze di Torino **119** (1985), 127–133.
9. Ji-Chang Kuang, *Applied Inequalities*, 2nd edition, Hunan Education Press, Changsha, China, 1993. (Chinese)
10. A. Laforgia, *Further inequalities for the gamma function*, Math. Comp. **42** (1984), 597–600.
11. A. Laforgia and S. Sismondi, *Monotonicity results and inequalities for the gamma and error functions*, J. Comp. Appl. Math. **23** (1988), 25–33.
12. E. Leach and M. Sholander, *Extended mean values*, Amer. Math. Monthly **85** (1978), 84–90.
13. M. Merkle, *Some inequalities for the chi squre distribution function*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **2** (1991), 89–94.
14. H. Minc and L. Sathre, *Some inequalities involving* $(r!)^{1/r}$, Proc. Edinburgh Math. Soc. **14** (1965/66), 41–46.
15. D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, New York/Heidelberg/Berlin, 1970.
16. D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.
17. B. Palumbo, *A generalization of some inequalities for the gamma function*, J. Comp. Appl. Math. **88** (1997), 255–268.
18. Feng Qi, *Generalized weighted mean values with two parameters*, Proc. Roy. Soc. London Ser. A **454** (1998), no. 1978, 2723–2732.
19. Feng Qi, *Generalized abstracted mean values*, RGMIA Research Report Collection **2** (1999), no. 5, Article 4. http://rgmia.vu.edu.au.
20. Feng Qi, *Logarithmic convexities of the extended mean values*, RGMIA Research Report Collection **2** (1999), no. 5, Article 5. http://rgmia.vu.edu.au.
21. Feng Qi, Li-Hong Cui, and Sen-Lin Xu, *Some inequalities constructed by Tchebysheff's integral inequality*, Mathematical Inequalities and Applications **2** (1999), no. 4, 517–528.
22. Feng Qi and Bai-Ni Guo, *Inequalities for generalized weighted mean values of convex function*, submitted for publication.
23. Feng Qi and Sen-Lin Guo, *Inequalities for the incomplete gamma and related functions*, Math. Ineq. Appl. **2** (1999), no. 1, 47–53.
24. Feng Qi and Qiu-Ming Luo, *A simple proof of monotonicity for extended mean values*, J. Math. Anal. Appl. **224** (1998), 356–359.
25. Feng Qi and Jia-Qiang Mei, *Some inequalities for the incomplete gamma and related functions*, Zeitschrift für Analysis und ihre Anwendungen **18** (1999), no. 3, 793–799.
26. Feng Qi, Sen-Lin Xu, and Lokenath Debnath, *A new proof of monotonicity for extended mean values*, International Journal of Mathematics and Mathematical Sciences **22** (1999), no. 2, 415–420.
27. Feng Qi and Shi-Qin Zhang, *Note on monotonicity of generalized weighted mean values*, Proceedings of the Royal Society of London Series A **455** (1999), no. 1989, 3259–3260.
28. H. R. van der Vaart, *Some extensions of the idea of Bias*, Ann. Math. Statist. **32** (1961), 436–447.
29. Feng Qi, Jia-Qiang Mei, and Sen-Lin Xu, *Other proofs of monotonicity for generalized weighted mean values*, RGMIA Research Report Collection **2** (1999), no. 4, Article 6. http://rgmia.vu.edu.au.

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