SEVERAL INTEGRAL INEQUALITIES

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Abstract. In the article, some integral inequalities are presented by analytic approach and mathematical induction. An open problem is proposed.

In this article, we establish some integral inequalities by analytic method and induction.

Proposition 1. *Let $f(x)$ be differentiable on (a,b) and $f(a)=0$. If $0 \leqslant f'(x) \leqslant 1$, then*

$$\int_a^b \bigl[f(x)\bigr]^3 \,\mathrm{d}x \leqslant \left(\int_a^b f(x)\,\mathrm{d}x\right)^2. \tag{1}$$

If $f'(x) \geqslant 1$, then inequality (1) *reverses. The equality in* (1) *holds only if $f(x) \equiv 0$ or $f(x) = x-a$.*

Proof. For $a \leqslant t \leqslant b$, set

$$F(t) = \left(\int_a^t f(x)\,\mathrm{d}x\right)^2 - \int_a^t \bigl[f(x)\bigr]^3 \,\mathrm{d}x.$$

Simple computation yields

$$\begin{aligned} F'(t) &= \left\{2\int_a^t f(x)\,\mathrm{d}x - \bigl[f(t)\bigr]^2\right\} f(t) \triangleq G(t)f(t),\\ G'(t) &= 2\bigl[1-f'(t)\bigr]f(t). \end{aligned}$$

Since $f'(t) \geqslant 0$ and $f(a)=0$, thus $f(t)$ is increasing and $f(t) \geqslant 0$.

(1) When $0 \leqslant f'(t) \leqslant 1$, we have $G'(t) \geqslant 0$, $G(t)$ increases and $G(t) \geqslant 0$ because of $G(a)=0$, hence $F'(t) = G(t)f(t) \geqslant 0$, $F(t)$ is increasing. Since $F(a)=0$, we have $F(t) \geqslant 0$, and $F(b) \geqslant 0$. Therefore, the inequality (1) holds.
(2) When $f'(t) \geqslant 1$, we have $G'(t) \leqslant 0$, $G(t)$ decreases, $G(t) \leqslant 0$, $F'(t) \leqslant 0$, and $F(t)$ is decreasing, then $F(t) \leqslant 0$, the inequality (1) reverses.
(3) Since the equality in (1) holds only if $f'(t)=1$ or $f(t)=0$, substitution of $f(t)=t+c$ into (1) and standard argument leads to $c=-a$.

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The proof is completed. □

Corollary 1 ([3, p. 624])**.** *Let $f(x)$ be a continuous function on the closed interval $[0,1]$ and $f(0)=0$, its derivative of the first order is bounded by $0\leqslant f'(x)\leqslant 1$ for $x\in(0,1)$. Then*

$$\int_0^1 \big[f(x)\big]^3\,\mathrm{d}x \leqslant \left(\int_0^1 f(x)\,\mathrm{d}x\right)^2. \tag{2}$$

Equality in (2) holds if and only if $f(x)=0$ or $f(x)=x$.

Proposition 2. *Suppose $f(x)$ has continuous derivative of the n-th order on the interval $[a,b]$, $f^{(i)}(a)\geqslant 0$ and $f^{(n)}(x)\geqslant n!$, where $0\leqslant i\leqslant n-1$, then*

$$\int_a^b \big[f(x)\big]^{n+2}\,\mathrm{d}x \geqslant \left(\int_a^b f(x)\,\mathrm{d}x\right)^{n+1}. \tag{3}$$

Proof. Let

$$H(t)=\int_a^t \big[f(x)\big]^{n+2}\,\mathrm{d}x-\left[\int_a^t f(x)\,\mathrm{d}x\right]^{n+1},\quad t\in[a,b]. \tag{4}$$

Direct calculation produces

$$\begin{aligned}
H'(t)=&\left\{\big[f(x)\big]^{n+1}-(n+1)\left[\int_a^t f(x)\,\mathrm{d}x\right]^{n}\right\}f(t)\triangleq h_1(t)f(t),\\
h_1'(t)=&(n+1)\left\{\big[f(x)\big]^{n-1}f'(t)-n\left[\int_a^t f(x)\,\mathrm{d}x\right]^{n-1}\right\}f(t)\triangleq (n+1)h_2(t)f(t),\\
h_2'(t)=&\left\{\big[f(x)\big]^{n-2}f''(t)+(n-1)\big[f(t)\big]^{n-3}\big[f'(t)\big]^2\right.\\
&\left.-n(n-1)\left[\int_a^t f(x)\,\mathrm{d}x\right]^{n-2}\right\}f(t)\triangleq h_3(t)f(t).
\end{aligned}$$

By induction, we obtain

$$h_i'(t)=\left\{f^{(i)}(t)\big[f(t)\big]^{n-i}+p_i(t)-\frac{n!}{(n-i)!}\left[\int_a^t f(x)\,\mathrm{d}x\right]^{n-i}\right\}f(t)\triangleq h_{i+1}(t)f(t), \tag{5}$$

where $2\leqslant i\leqslant n$ and

$$\begin{aligned}
p_2(t)&=(n-1)\big[f(t)\big]^{n-3}\big[f'(t)\big]^2,\\
p_{i+1}(t)f(t)&=p_i'(t)+(n-i)f^{(i)}(t)\big[f(t)\big]^{n-i-1}f'(t).
\end{aligned} \tag{6}$$

¿From $f^{(n)}(t)\geqslant n!$ and $f^{(i)}(a)\geqslant 0$ for $0\leqslant i\leqslant n-1$, it follows that $f^{(i)}(t)\geqslant 0$ and are increasing for $0\leqslant i\leqslant n-1$.

Using the mathematical induction, it is easy to see that

$$p_i(t)=\sum_{j_0+\sum_{k=1}^{i-1}k\cdot j_k=n-1} C(j_0,j_1,\dots,j_{i-1})\prod_{k=0}^{i-1}\big[f^{(k)}(t)\big]^{j_k},$$

where j_k and $C(j_0, j_1, \dots, j_{i-1})$ are nonnegative integers, $0 \leqslant k \leqslant i-1$.

Therefore, we obtain $p'_k(t) \geqslant 0$ and $p_{k+1}(t) \geqslant 0$, then $p'_{k-1}(t)$ and $p_k(t)$ are increasing for $2 \leqslant k \leqslant n$. Straightforward computation yields

$$h_{n+1}(t) = f^{(n)}(t) + p_n(t) - n!.$$

Considering $f^{(n)}(t) \geqslant n!$, we get $h_{n+1}(t) \geqslant 0$, and $h'_n(t) \geqslant 0$, then $h_n(t)$ increases.

By definitions of $h_i(t)$, we have, for $1 \leqslant i \leqslant n-1$,

$$h_{i+1}(a) = f^{(i)}(a)\big[f(a)\big]^{n-i} + p_i(a) \geqslant 0.$$

Therefore, using induction on i, we obtain $h'_i(t) \geqslant 0$, $h_i(t) \geqslant 0$, and $h_i(t)$ are increasing for $1 \leqslant i \leqslant n$. Then $H'(t) \geqslant 0$ and increases, and $H(t) \geqslant 0$. The inequality (3) follows from $H(b) \geqslant 0$. The proposition 2 is proved. □

Corollary 2. *Let $f(x)$ be n-times differentiable on $[a, b]$, $f^{(i)}(a) \geqslant 0$ and $f^{(n)}(x) \geqslant n!$ for $0 \leqslant i \leqslant n-1$. Then the functions $H(t)$, $h_j(t)$ and $p_k(t)$ defined by the formulae* (4), (5) *and* (6) *are increasing and convex, where $1 \leqslant j \leqslant n-1$ and $2 \leqslant k \leqslant n-2$.*

Remark. The inequality (3) is not found in [1, 2, 4, 5]. So maybe it is a new inequality.

At last, we propose the following open problem in a natural way:

Open Problem. *Under what conditions does the inequality*

$$\int_a^b \big[f(x)\big]^t \,\mathrm{d}x \geqslant \left(\int_a^b f(x)\,\mathrm{d}x\right)^{t-1} \tag{7}$$

hold for $t > 1$?

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