CHARACTERIZATIONS OF STABILITY FOR STRONGLY CONTINUOUS SEMIGROUPS BY CONVOLUTIONS

C. Buşe, S. S. Dragomir and **V. Lupulescu**

Abstract

Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup of bounded linear operators acting on a Banach space X. We prove that if the convolution $\mathbf{T} * (e^{-i\mu(\cdot)}f)$ is bounded for every continuous and 1-periodic function which is null in $t = 0$ and some $\mu \in \mathbf{R}$, then $T(1)$ is power bounded and $e^{i\mu} \in \rho(T(1))$. Applications to questions of exponential stability are also presented.

1. Introduction A well-known result of M.G. Krein, see [K] or [DK], says:

Let X be a Banach space and A be a linear and bounded operator acting on X. If for all $\mu \in \mathbf{R}$ and every $x_0 \in X$ the solution of the Cauchy problem

$$\dot{x}(t) = Ax(t) + e^{i\mu t}x_0, \quad x(0) = 0, \quad t \geq 0$$

is bounded, then there exists constants $N > 0$ and $\nu > 0$ such that

$$||e^{tA}|| \leq Ne^{-\nu t}, \forall t \geq 0.$$

A proof of this classic result can be found in [Ba]. The above result cannot be extended for the case when A is the infinitesimal generator of a strong semigroup, cf [RB, Example 3.1]. However, weakly related results, hold. For example, in [VuS] (Corollary 4.5 and its reformulation), it has been proved that if the Cauchy problem

$$\dot{x}(t) = Ax(t) + f(t), \quad t \geq 0, \quad x(0) = 0 \qquad (A, f, 0)$$

has a bounded solution for every $f \in \mathcal{P}(\omega)$ (that is, f is a continuous and ω-periodic function) then $1 \in \rho(T(\omega))$. Moreover the semigroup $\mathbf{T}$ is uniformly exponentially stable if and only if for every $f \in BUC(\mathbf{R}_+, X)$ (or $f \in AP(\mathbf{R}_+, X)$) the solution of the problem $(A, f, 0)$, is bounded. A short history about the problematic exposed previously and other references can be found in [VuS]. We present in the following simple generalizations of the above results. Our proofs are elementary and only use first principles.

2. Preliminary results. Let X be a real or complex Banach space and $L(X)$ the Banach algebra of all linear and bounded operators acting on X. We denote by $||\cdot||$, the norms of vectors and operators. Let $T \in L(X)$. We will denote by $\sigma(T)$ the spectrum of T and with $r(T)$ we denote the spectral radius of T. We recall that

$$r(T) = \sup\{|z| : z \in \sigma(T)\}. \tag{1}$$

The resolvent set of T is $\rho(T) := \mathbf{C} \setminus \sigma(T)$, i.e. the set of all complex scalar λ such that $\lambda Id - T$ is an invertible operator. Id denotes here the identity operator in $L(X)$. We recall that an operator $T \in L(X)$ is power bounded if there exists an $M > 0$ such that

$$||T^n|| \leq M, \quad \forall n \in \mathbf{N} = \{0, 1, 2, \cdots\}.$$

We shall prove several lemmas which would be used later.

Lemma 1 *Let $T \in L(X)$. If there exist $M > 0$ such that*

$$\sup\{||Id + T + \cdots + T^n|| : \quad n \in \{1, 2, \cdots\}\} = M < \infty \tag{2}$$

then T is power bounded and $1 \in \rho(T)$.

Proof. The first assertion follows from (2) and the identity

$$T^{n+1} = Id + (T - Id)(Id + T + \cdots + T^n).$$

Suppose that $1 \in \sigma(T)$. Then there exists a sequence $(x_m)_{m \in \mathbf{N}}$ with $x_m \in X, ||x_m|| = 1$ and $(Id - T)x_m \to 0$ as $m \to \infty$, (see [Na, Proposition 2.2, p. 64]). However, T is power bounded, and hence $T^k(Id - T)x_m \to 0$ as $m \to \infty$, uniformly for $k \in \mathbf{N}$. Let $N \in \mathbf{N}$, $N > 2M$ and $m \in \mathbf{N}$ such that

$$||T^k(Id - T)x_m|| \leq \frac{1}{2N}, \quad k = 0, 1, \cdots N.$$

Then

$$\begin{aligned} M &\geq \|x_m + \sum_{k=1}^{N}(x_m + \sum_{j=0}^{k-1} T^j(T - Id)x_m)\| \\ &= \|(N+1)x_m + \sum_{k=1}^{N}\sum_{j=0}^{k-1} T^j(T - Id)x_m\| \\ &\geq (N+1) - \tfrac{N(N+1)}{4N} > \tfrac{N}{2} > M. \end{aligned}$$

This is a contradiction and thus $1 \notin \sigma(T)$.

Lemma 2 *Let $U \in L(X)$ and $\mu \in \mathbf{R}$. If*

$$\sup_{n=1,2,\cdots}\{||\sum_{k=0}^{n} e^{i\mu k}U^k||\} = M_\mu < \infty \tag{3}$$

then U is power bounded and $e^{-i\mu} \in \rho(U)$.

Proof. It follows from Lemma 1 for $T = e^{i\mu}U$.

Lemma 3 *Let $U \in L(X)$. If the condition (3) holds for all $\mu \in \mathbf{R}$, then $r(T) < 1$.*

Proof. It follows by (1), Lemma 2 and the fact that $\sigma(T)$ is a compact set.

3. Exponential stability and convolutions

We recall that a strongly continuous semigroup is a family $\mathbf{T} = \{T(t)\}_{t\geq 0}$ of bounded linear operators acting on the Banach space X which satisfies the following conditions:

- **(i)** $T(t+s) = T(t)T(s)$ for all $t, s \in \mathbf{R}_+ := [0, \infty)$;
- **(ii)** $T(0) = Id$;
- **(iii)** the function $t \mapsto T(t)x : \mathbf{R}_+ \to X$ is continuous on $\mathbf{R}_+$ for all $x \in X$.

The semigroups theory is developed in the books [Pa], [vC], [Na], [Ne] and others.

Let $P_1^0(\mathbf{R}_+, X)$ be the set of all continuous X-valued functions such that $f(t+1) = f(t)$ for all $t \geq 0$ and $f(0) = 0$.

Proposition 4 *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on X and $\mu \in \mathbf{R}$. If*

$$\sup_{t>0} ||\int_0^t e^{i\mu\xi}T(t-\xi)f(\xi)d\xi|| < \infty,$$

for all $f \in P_1^0(\mathbf{R}_+, X)$, (4)

then $T(1)$ is power bounded and $e^{i\mu} \in \rho(T(1))$.

Proof. Let $U = T(1))$, $x \in X$ and $f_1 \in P_1^0(\mathbf{R}_+, X)$, the function defined by

$$f_1(\xi) = \xi(1-\xi)T(\xi)x,$$

for all $\xi \in [0,1]$. From (4), it follows that

$$\sup_{n\in\{1,2,\cdots\}} \|\sum_{k=0}^{n} \int_{k}^{k+1} T(n+1-\xi)e^{-i\mu\xi} f_1(\xi)d\xi\| = M(\mu, f_1) < \infty. \qquad (5)$$

Simple calculus gives

$$\int_{k}^{(k+1)} T(n+1-\xi)e^{-i\mu\xi} f_1(\xi)d\xi$$

$$= e^{i\mu(n+1)}(\int_0^1 e^{-i\mu\xi}\xi(1-\xi)d\xi)e^{-i\mu(n-k+1)}T(n-k+1)x.$$

Substituting this into (4), we obtain:

$$\sup_{n\in\mathbf{N}} \|\sum_{j=1}^{n+1} e^{-\mu j}U^j\| < \infty.$$

Now, from Lemma 2, it follows that $T(1)$ is power bounded and $e^{i\mu} \in \rho(T(1))$.

Corollary 5 *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on the Banach space X. If the condition (4) holds for all $\mu \in \mathbf{R}$ and every $f \in P_1^0(\mathbf{R}_+, X)$, then $r(T(1)) < 1$ and $\mathbf{T}$ is uniformly exponentially stable.*

Proof. We recall that a strongly continuous semigroup on X is uniformly exponentially stable if its growth bound $\omega_0(\mathbf{T})$ is negative (or, equivalently) if there exists the constants $N > 0$ and $\nu > 0$ such that

$$\|T(t)\| \leq Ne^{-\nu t} \quad \forall t \geq 0.$$

The above assertion follows from Proposition 4, Lemma 3 and the fact that $r(T(1)) = e^{\omega_0(\mathbf{T})}$, cf. [Ne, Proposition 1.2.2].

Let $BUC(\mathbf{R}_+, X)$ the Banach space of all X-valued, bounded and uniformly continuous functions on $\mathbf{R}_+$, endowed with the sup-norm and $AP(\mathbf{R}_+, X)$

the space of almost periodic functions in the sense of Bohr, i.e. $AP(\mathbf{R}_+, X)$ is the linear closed hull in $BUC(\mathbf{R}_+, X)$ of the set of all functions:

$$\{e^{i\mu(\cdot)}x : \quad \mu \in \mathbf{R} \quad x \in X\}.$$

Let $AP_0(\mathbf{R}_+, X)$ be the set of all $f \in AP(\mathbf{R}_+, X)$ such that $f(0) = 0$. It is clear that $AP_0(\mathbf{R}_+, X)$ is a closed subspace of either $AP(\mathbf{R}_+, X)$ or $BUC(\mathbf{R}_+, X)$.

Corollary 6 *Let* $\mathbf{T} = \{T(t)\}_{t\geq 0}$ *be a strongly continuous semigroup on* X. *If*

$$\sup_{t\geq 0} || \int_0^t T(\xi)f(t-\xi)d\xi|| < \infty, \quad \$$$

for all $f \in AP_0(\mathbf{R}_+, X)$ *then* $\mathbf{T}$ *is uniformly exponentially stable.*

References

[Ba] Balint, S., *On the Perron-Bellman theorem for systems with constants cofficients*, Ann. Univ. Timişoara, **21**, fasc. 1-2(1983), 3-8.

[vC] J. Van Casteren, *Generators of Strongly Continuous Semigroups*, Pitman, Boston-London-Melbourne, 1985.

[DK] Daletckii Ju.L. and Krein M.G., *Stability of Solutions of Differential Equations in Banach Spaces*, Amm. Math. Soc. Providence , RI, 1974

[K] M. G. Krein, *On some questions related to the ideas of Lyapunov in the theory of stability*, Uspeckhi Mat. Nauk **3**, 1948, 166-169, [in Russian].

[Na] Nagel R. (ed.), *One-parameter semigroups of positive operators,* Lect. Notes in Math. 1184, Springer- Verlag, Berlin-New York, 1986.

[Ne] J.M.A.M. van Neerven, *The Asymptotic Behaviour of Semigroups of Linear Operators*, Birkhäuser-Verlag, 1996

[Pa] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer- Verlag, 1983.

[VuS] Vũ Quôc Phóng and E. Schüler, *The operator equation* $AX - XB = C$*, admisibility and asymptotic behaviour of differential equations*, J. Diff. Equations, **145** (1998), 394-419.

[RB] Reghiş M., and Buşe C., *On the Perron-Bellman theorem for* C_0*-semigroups and periodic evolutionary processes in Banach spaces,* Italian Journal of Pure and Applied Mathematics, **4**(1998), 155-166.