INEQUALITIES FOR GENERALIZED WEIGHTED MEAN VALUES OF CONVEX FUNCTION

BAI-NI GUO AND FENG QI

Abstract. In the article, using the Tchebycheff's integral inequality, the suitable properties of double integral and the Cauchy's mean value theorem in integral form, the following result is proved: Suppose $f(x)$ is a positive differentiable function and $p(x) \not\equiv 0$ an integrable nonnegative weight on the interval $[a,b]$, if $f'(x)$ and $f'(x)/p(x)$ are integrable and both increasing or both decreasing, then for all real numbers r and s, we have

$$M_{p,f}(r,s;a,b) < E\big(r+1,s+1;f(a),f(b)\big); \tag{$*$}$$

if one of the functions $f'(x)$ or $f'(x)/p(x)$ is nondecreasing and the other nonincreasing, then the inequality $(*)$ reverses. Where $M_{p,f}(r,s;a,b)$ and $E(r,s;a,b)$ denote the generalized weighted mean values of function f with two parameters r, s and weight p and the extended mean values, respectively. This inequality $(*)$ generalizes the Hermite-Hadamard's inequality, and the like.

1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be a convex function, then

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \le \frac{f(a)+f(b)}{2}. \tag{1}$$

The inequality (1) is called Hermite-Hadamard's inequality in [4] and [5, pp. 10–12]. The middle term of inequality (1) is called the arithmetic mean of the function $f(x)$ on the interval $[a,b]$, the right term in (1) is the arithmetic mean of numbers $f(a)$ and $f(b)$.

Let $f(x)$ be a positive integrable function on the interval $[a,b]$, then the power mean of $f(x)$ is defined as follows

$$M_\alpha(f) = \begin{cases} \left(\dfrac{\int_a^b f^\alpha(x)\mathrm{d}x}{b-a}\right)^{1/\alpha}, & \alpha \ne 0,\\ \exp\left(\dfrac{\int_a^b \ln f(x)\mathrm{d}x}{b-a}\right), & \alpha = 0. \end{cases} \tag{2}$$

The generalized logarithmic mean (or Stolarsky's mean) on the interval $[a,b]$ is defined for $x>0$, $y>0$ by

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$$
(3)\qquad S_\alpha(x,y)=\begin{cases}\left(\dfrac{x^\alpha-y^\alpha}{\alpha(x-y)}\right)^{1/(\alpha-1)}, & \alpha\neq 0,1\quad x-y\neq 0;\\ \dfrac{y-x}{\ln y-\ln x}, & \alpha=0,\quad x-y\neq 0;\\ \dfrac{1}{e}\left(\dfrac{x^x}{y^y}\right)^{1/(x-y)}, & \alpha=1,\quad x-y\neq 0;\\ x, & x-y=0.\end{cases}
$$

In [5, p. 12] and [13], Zhen-Hang Yang has given the following generalizations of the Hermite-Hadamard's inequality (1):

If $f(x)>0$ has derivative of second order and $f''(x)>0$, for $\lambda>1$, we have

(i) $f^\lambda\Big(\dfrac{a+b}{2}\Big)<\dfrac{1}{b-a}\displaystyle\int_a^b f^\lambda(x)\mathrm{d}x<\dfrac{f^\lambda(a)+f^\lambda(b)}{2}$;

(ii) $f\Big(\dfrac{a+b}{2}\Big)<M_\lambda(f)<N_\lambda(f(a),f(b))$, where

$$
N_\lambda(x,y)=\begin{cases}\dfrac{x^\lambda+y^\lambda}{2}, & \lambda\neq 0,\\ \sqrt{xy}, & \lambda=0;\end{cases}
$$

(iii) For all real number α, $M_\alpha(f)<S_{\alpha+1}\big(f(a),f(b)\big)$;

(iv) For $\alpha\geq 1$, $f\Big(\dfrac{a+b}{2}\Big)<M_\alpha(f)<S_{\alpha+1}\big(f(a),f(b)\big)$.

(v) If $f''(x)<0$ for $x\in(a,b)$, the above inequalities are all reversed.

In [11], two-parameter mean is defined as

$$
(4)\qquad M_{p,q}(f)=\begin{cases}\left(\dfrac{\int_a^b f^p(x)\mathrm{d}x}{\int_a^b f^q(x)\mathrm{d}x}\right)^{1/(p-q)}, & p\neq q,\\ \exp\left(\dfrac{\int_a^b f^p(x)\ln f(x)\mathrm{d}x}{\int_a^b f^p(x)\mathrm{d}x}\right), & p=q.\end{cases}
$$

When $q=0$, $M_{p,0}(f)=M_p(f)$; when $f(x)=x$, the two-parameter mean is reduced to the extended mean values $E(r,s;x,y)$ for positive x and y:

$$
(5)\qquad E(r,s;x,y)=\left[\frac{r}{s}\cdot\frac{y^s-x^s}{y^r-x^r}\right]^{1/(s-r)},\qquad rs(r-s)(x-y)\neq 0;
$$

$$
(6)\qquad E(r,0;x,y)=\left[\frac{1}{r}\cdot\frac{y^r-x^r}{\ln y-\ln x}\right]^{1/r},\qquad r(x-y)\neq 0;
$$

$$
(7)\qquad E(r,r;x,y)=\mathrm{e}^{-1/r}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)},\qquad r(x-y)\neq 0;
$$

$$
E(0,0;x,y)=\sqrt{xy},\qquad x\neq y;
$$

$$
E(r,s;x,x)=x,\qquad x=y.
$$

In 1997, Ming-Bao Sun [11] generalized Hermite-Hadamard's inequality (1) and the results derived by Yang in [5, 13] to obtain that, if the positive function $f(x)$

has derivative of second order and $f''(x) > 0$, then, for all real numbers p and q,

$$M_{p,q}(f) < E\big(p+1, q+1; f(a), f(b)\big). \tag{8}$$

If $f''(x) < 0$, then inequality (8) is reversed.

Recently the first author established in [7, 8] the generalized weighted mean values $M_{p,f}(r,s;x,y)$ of a positive function f defined on the interval between x and y with two parameters $r, s \in \mathbb{R}$ and nonnegative weight $p \not\equiv 0$ by

$$\begin{aligned} M_{p,f}(r,s;x,y) &= \left(\frac{\int_x^y p(u) f^s(u) \mathrm{d}u}{\int_x^y p(u) f^r(u) \mathrm{d}u}\right)^{1/(s-r)}, && (r-s)(x-y) \neq 0; \\ M_{p,f}(r,r;x,y) &= \exp\left(\frac{\int_x^y p(u) f^r(u) \ln f(u) \mathrm{d}u}{\int_x^y p(u) f^r(u) \mathrm{d}u}\right), && x-y \neq 0; \\ M_{p,f}(r,s;x,x) &= f(x), && x=y. \end{aligned} \tag{9}$$

It is well-known that the concepts of means and their inequalities not only are basic and important concepts in mathematics (for example, some definitions of norms are often special means) and have explicit geometric meanings [10], but also have applications in electrostatics [6], heat conduction and chemistry [12]. Moreover, some applications to medicine are given in [1].

In this article, using the Tchebycheff's integral inequality, suitable properties of double integral and the Cauchy's mean value theorem in integral form, the following result is obtained:

Main Theorem. *Suppose $f(x)$ is a positive differentiable function and $p(x) \not\equiv 0$ an integrable nonnegative weight on the interval $[a,b]$, if $f'(x)$ and $f'(x)/p(x)$ are both increasing or both decreasing and integrable, then for all real numbers r and s, we have*

$$M_{p,f}(r,s;a,b) < E\big(r+1, s+1; f(a), f(b)\big); \tag{$*$}$$

if one of the functions $f'(x)$ or $f'(x)/p(x)$ is nondecreasing and the other nonincreasing, then the inequality $(*)$ reverses.

2. Proof of Main Theorem

In order to verify the Main Theorem, the following lemmas are necessary.

Lemma 1. *Let $G, H : [a,b] \to \mathbb{R}$ be integrable functions, both increasing or both decreasing. Furthermore, let $Q : [a,b] \to [0,+\infty)$ be an integrable function. Then*

$$\int_a^b Q(u)G(u)\mathrm{d}u \int_a^b Q(u)H(u)\mathrm{d}u \leq \int_a^b Q(u)\mathrm{d}u \int_a^b Q(u)G(u)H(u)\mathrm{d}u, \tag{10}$$

with equality if and only if one of the functions G and H reduces to a constant.

If one of the functions of G or H is nonincreasing and the other nondecreasing, then the inequality (10) reverses.

Inequality (10) is called the Tchebycheff's integral inequality [2, 5].

Lemma 2 ([9])**.** *Suppose that $f(t)$ and $g(t) \geq 0$ are integrable on $[a,b]$ and the ratio $f(t)/g(t)$ has finitely many removable discontinuity points. Then there exists at least one point $\theta \in (a,b)$ such that*

$$\frac{\int_a^b f(t)\mathrm{d}t}{\int_a^b g(t)\mathrm{d}t} = \lim_{t\to\theta} \frac{f(t)}{g(t)}. \tag{11}$$

We call Lemma 2 the revised Cauchy's mean value theorem in integral form.

Proof. Since $f(t)/g(t)$ has finitely many removable discontinuity points, without loss of generality, suppose it is continuous on $[a, b]$. Furthermore, using $g(t) \geq 0$, from the mean value theorem for integrals in standard textbook of mathematical analysis or calculus, there exists at least one point $\theta \in (a, b)$ satisfying

$$\int_a^b f(t)\mathrm{d}t = \int_a^b \Big(\frac{f(t)}{g(t)}\Big) g(t)\mathrm{d}t = \frac{f(\theta)}{g(\theta)} \int_a^b g(t)\mathrm{d}t. \tag{12}$$

Lemma 2 follows. □

Proof of Main Theorem. It is sufficient to prove the Main Theorem only for $s > r$ and for $f'(x)$ and $f'(x)/p(x)$ both being increasing. The remaining cases can be done similarly.

Case 1. When $s > r$ and $f(a) \neq f(b)$, inequality $(*)$ is equivalent to

$$\int_a^b p(x)f^s(x)\mathrm{d}x \left| \int_a^b f^r(x)f'(x)\mathrm{d}x \right| < \int_a^b p(x)f^r(x)\mathrm{d}x \left| \int_a^b f^s(x)f'(x)\mathrm{d}x \right|. \tag{13}$$

Take $G(x) = f^{s-r}(x), H(x) = f'(x)/p(x)$ (being increasing) and $Q(x) = p(x)f^r(x) \geq 0$ in inequality (10). If $f'(x) > 0$, then $f^{s-r}(x)$ is increasing, inequality (13) holds. If $f'(x) < 0$, then $f^{s-r}(x)$ decreases, inequality (13) is still valid.

If $f'(x)$ does not keep the same sign on (a, b), then there exists an unique point $\theta \in (a, b)$ such that $f'(x) > 0$ on (θ, b) and $f'(x) < 0$ on (a, θ). Further, if $f(a) < f(b)$, then there exists an unique point $\xi \in (\theta, b)$ such that $f(\xi) = f(a)$. Therefore, inequality (13) is also equivalent to

$$\int_a^b p(x)f^s(x)\mathrm{d}x \int_\xi^b f^r(x)f'(x)\mathrm{d}x < \int_a^b p(x)f^r(x)\mathrm{d}x \int_\xi^b f^s(x)f'(x)\mathrm{d}x. \tag{14}$$

Using inequality (10) again produces

$$\int_\xi^b p(x)f^s(x)\mathrm{d}x \int_\xi^b f^r(x)f'(x)\mathrm{d}x < \int_\xi^b p(x)f^r(x)\mathrm{d}x \int_\xi^b f^s(x)f'(x)\mathrm{d}x. \tag{15}$$

For $x \in (a, \xi), y \in (\xi, b)$, we have $f'(y) > 0, f(x) < f(a) = f(\xi) < f(y)$ and $f^{s-r}(x) < f^{s-r}(y)$, therefore, suitable properties of double integral leads to

$$\begin{aligned} &\int_a^\xi p(x)f^s(x)\mathrm{d}x \int_\xi^b f^r(x)f'(x)\mathrm{d}x - \int_a^\xi p(x)f^r(x)\mathrm{d}x \int_\xi^b f^s(x)f'(x)\mathrm{d}x \\ &= \iint\limits_{[a,\xi]\times[\xi,b]} p(x)f^r(x)f^r(y)f'(y)\big[f^{s-r}(x) - f^{s-r}(y)\big]\mathrm{d}x\mathrm{d}y < 0. \end{aligned} \tag{16}$$

¿From this, we conclude that inequality (14) is valid, namely, inequality (13) holds.

If $f'(x)$ does not keep the same sign on (a, b) and $f(b) < f(a)$, from the same arguments as the case of $f(b) > f(a)$, inequality (13) follows.

Case 2. When $s > r$ and $f(a) = f(b)$, since $f'(x)$ increases, we have $f(x) < f(a) = f(b), x \in (a, b)$. From the definition of $E(r, s; x, y)$, inequality $(*)$ is equivalent to

$$M_{p,f}(r, s; a, b) < f(a) = f(b), \tag{17}$$

that is

$$\frac{\int_a^b p(x)f^s(x)\mathrm{d}x}{\int_a^b p(x)f^r(x)\mathrm{d}x} < f^{s-r}(a) = f^{s-r}(b). \tag{18}$$

This follows from Lemma 2.

The proof of Main Theorem is completed. □

3. Applications

It is well-known that mean S_0 is called the logarithmic mean denoted by L, and S_1 the identric mean or the exponential mean by I.

The logarithmic mean $L(x, y)$ can be generalized to the one-parameter means:

$$\begin{aligned} &J_p(x,y) = \frac{p(y^{p+1} - x^{p+1})}{(p+1)(y^p - x^p)}, \quad x \neq y, \quad p \neq 0, -1; \\ &J_0(x,y) = L(x,y), \quad J_{-1}(x,y) = \frac{G^2}{L}; \\ &J_p(x,x) = x. \end{aligned} \tag{19}$$

Here, $J_{1/2}(x, y) = h(x, y)$ is called the Heron's mean and $J_2(x, y) = c(x, y)$ the centroidal mean. Moreover, $J_{-2}(x, y) = H(x, y)$, $J_1(x, y) = A(x, y)$, $J_{-1/2}(x, y) = G(x, y)$.

The extended Heron's means $h_n(x, y)$ is defined by

$$h_n(x,y) = \frac{1}{n+1} \cdot \frac{x^{1+1/n} - y^{1+1/n}}{x^{1/n} - y^{1/n}}. \tag{20}$$

Let f and p be defined and integrable functions on the closed interval $[a, b]$. The weighted mean $M^{[r]}(f; p; x, y)$ of order r of the function f on $[a, b]$ with the weight p is defined [3, pp. 75–76] by

$$M^{[r]}(f;p;x,y) = \begin{cases} \left(\dfrac{\int_x^y p(t)f^r(t)\mathrm{d}t}{\int_x^y p(t)\mathrm{d}t} \right)^{1/r}, & r \neq 0; \\ \exp\left(\dfrac{\int_x^y p(t)\ln f(t)\mathrm{d}t}{\int_x^y p(t)\mathrm{d}t} \right), & r = 0. \end{cases} \tag{21}$$

It is clear that $M^{[r]}(f; p; x, y) = M_{p,f}(r, 0; x, y)$, $E(r, s; x, y) = M_{1,x}(r-1, s-1; x, y)$, $E(r, r+1; x, y) = J_r(x, y)$. From these definitions of mean values and some relationships between them, we can easily get the following inequalities:

Corollary. *Let $f(x)$ be a positive differentiable function and $p(x) \not\equiv 0$ an integrable nonnegative weight on the interval $[a, b]$. If $f'(x)$ and $f'(x)/p(x)$ are integrable and* *both increasing or both decreasing, then for all real numbers r, s, we have*

(22) $M^{[r]}(f;p;a,b) < S_{r+1}\big(f(a),f(b)\big), \qquad M_{p,f}(0,-1;a,b) < L\big(f(a),f(b)\big),$

(23) $M_{p,f}(0,0;a,b) < I\big(f(a),f(b)\big), \qquad M_{p,f}(0,1;a,b) < A\big(f(a),f(b)\big),$

(24)
$$M_{p,f}(-1,-1;a,b) < G\big(f(a),f(b)\big), \qquad M_{p,f}(-3,-2;a,b) < H\big(f(a),f(b)\big),$$

(25)
$$M_{p,f}\Big(\frac{1}{2},-\frac{1}{2};a,b\Big) < h\big(f(a),f(b)\big), \qquad M_{p,f}\Big(\frac{1}{n},\frac{1}{n}-1;a,b\Big) < h_n\big(f(a),f(b)\big),$$

(26) $M_{p,f}(2,1;a,b) < c\big(f(a),f(b)\big), \qquad M_{p,f}(-1,-2;a,b) < \dfrac{G^2\big(f(a),f(b)\big)}{L\big(f(a),f(b)\big)},$

(27)
$$M_{p,f}(r,r+1;a,b) < J_r\big(f(a),f(b)\big).$$

If one of the functions $f'(x)$ or $f'(x)/p(x)$ is nondecreasing and the other nonincreasing, then all of the inequalities from (22) *to* (27) *reverse.*

Remark 1. If take $p(x) \equiv 1$ and special values of r and s in Main Theorem or Corollary, we can derive the Hermite-Hadamard's inequality (1) and all of the related inequalities in [5, 11, 13], and the like.

Remark 2. The mean $M_{p,f}(0,1;a,b)$ is called the weighted arithmetic mean, $M_{p,f}(-1,-1;a,b)$ the weighted geometric mean, $M_{p,f}(-3,-2;a,b)$ the weighted harmonic mean of the function $f(x)$ on the interval $[a,b]$ with weight $p(x)$, respectively. So, we can seemingly call $M^{[r]}(f;p;a,b)$, $M_{p,f}(0,-1;a,b)$, $M_{p,f}(0,0;a,b)$, $M_{p,f}(\frac{1}{n},\frac{1}{n}-1;a,b)$ and $M_{p,f}(1,2;a,b)$ the weighted Stolarsky's (or generalized logarithmic) mean, the weighted logarithmic mean, the weighted exponential mean, the weighted Heron's mean and the weighted centroidal mean of the function $f(x)$ on the interval $[a,b]$ with weight $p(x)$, respectively.

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Department of Mathematics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, The People's Republic of China
E-mail address: qifeng@jzit.edu.cn