BETTER BOUNDS FOR AN INEQUALITY OF THE OSTROWSKI TYPE WITH APPLICATIONS

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ABSTRACT. In this paper we improve a recent result by Matić, Pečarić and Ujević [6] and apply it for special means and cumulative probability functions.

1. Introduction

In 1938, A. Ostrowski [1, p. 468] proved the following inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) M \tag{1.1}$$

for all $x \in [a,b]$, provided that f is differentiable on (a,b) and $|f'(t)| \le M$ for all $t \in (a,b)$.

Using the following representation, which has been obtained by Montgomery in an equivalent form [1, p. 565]

$$f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt = \frac{1}{b-a} \int_a^b p(x,t) f'(t)\,dt \tag{1.2}$$

for all $x \in [a,b]$, provided that f is absolutely continuous on $[a,b]$ and

$$p(x,t) := \begin{cases} t-a & \text{if } \quad t \in [a,x] \\ \\ t-b & \text{if } \quad t \in (x,b\} \end{cases}, (x,t) \in [a,b]^2,$$

we can put in place of M, i.e., in (1.1), the sup norm of f', i.e., $\|f'\|_\infty$ where

$$\|f'\|_\infty := ess \sup_{t \in [a,b]} |f'(t)|,$$

provided that $f' \in L_\infty[a,b]$.

For other Ostrowski type inequalities for mappings of bounded variation, monotonic or Lipschitzian, or generalisations for $n-$time differentiable mappings, see the book [1], the paper [2] by A.M. Fink, or the recent papers [3]-[4]. For on-line access to some related results in preprint, visit the website address
`http://rgmia.vu.edu.au/IneqNumAnaly.html`

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In [5], Dragomir and Wang, by the use of the Grüss inequality, proved the following perturbed version of Ostrowski's inequality:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right| \leq \frac{1}{4}(b-a)(\Gamma-\gamma) \tag{1.3}$$

for all $x\in[a,b]$, provided the derivative f' satisfies the condition

$$\gamma\leq f'(t)\leq\Gamma \quad \text{on} \quad (a,b). \tag{1.4}$$

Using a pre-Grüss inequality, Matić, Pečaric and Ujević [6] improved the constant $\frac{1}{4}$, in the right hand member of (1.3), with the constant $\frac{1}{4\sqrt{3}}$.

For some generalisations of (1.3), see [7] by Fedotov and Dragomir.

An upper bound in terms of the second derivative has been pointed out by Barnett and Dragomir in [8].

For two mappings $g,h:[a,b]\to\mathbb{R}$, define the *Chebychev functional* as

$$T(g,h):=\frac{1}{b-a}\int_a^b g(t)\,h(t)\,dt-\frac{1}{b-a}\int_a^b g(t)\,dt\cdot\frac{1}{b-a}\int_a^b h(t)\,dt,$$

provided the involved integrals exist.

In this note, by the use of Chebychev's functional, we improve the Matić-Pečaric-Ujević result by providing a better bound for the first membership of (1.3) in terms of Euclidean norms. Since the bound in (1.3) will apply for absolutely continuous mappings whose derivatives are bounded, the new inequality will also apply for the larger class of absolutely continuous mappings whose derivative $f'\in L_2[a,b]$. Some applications for special means and probability density functions are also given.

2. The Results

The following theorem holds.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping whose derivative $f'\in L_2[a,b]$. Then we have the inequality*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right| \\ &\leq \frac{(b-a)}{2\sqrt{3}}\left[\frac{1}{b-a}\|f'\|_2^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}} \\ &\quad\left(\leq\frac{(b-a)(\Gamma-\gamma)}{4\sqrt{3}} \quad \text{if} \quad \gamma\leq f'(t)\leq\Gamma \quad \text{for a.e. } t \text{ on } [a,b]\right)\end{aligned} \tag{2.1}$$

for all $x\in[a,b]$.

Proof. We use Korkine's identity:

$$T(g,h):=\frac{1}{2(b-a)^2}\int_a^b\int_a^b(g(t)-g(s))(h(t)-h(s))\,dtds,$$

to obtain

$$\frac{1}{b-a}\int_a^b p(x,t) f'(t)\,dt - \frac{1}{b-a}\int_a^b p(x,t)\,dt \cdot \frac{1}{b-a}\int_a^b f'(t)\,dt \tag{2.2}$$
$$= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))(f'(t)-f'(s))\,dtds.$$

As

$$\begin{aligned} \frac{1}{b-a}\int_a^b p(x,t) f'(t)\,dt &= f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt, \\ \frac{1}{b-a}\int_a^b p(x,t)\,dt &= x - \frac{a+b}{2} \end{aligned}$$

and

$$\frac{1}{b-a}\int_a^b f'(t)\,dt = \frac{f(b)-f(a)}{b-a},$$

then, by (2.2), we get the following identity which is of interest in its own right.

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) \tag{2.3}$$
$$= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))(f'(t)-f'(s))\,dtds$$

for all $x \in [a,b]$.

Using the Cauchy-Buniakowski-Schwartz inequality for double integrals, we may write

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))(f'(t)-f'(s))\,dtds \tag{2.4}$$
$$\leq \left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))^2\,dtds\right)^{\frac{1}{2}}$$
$$\times \left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b (f'(t)-f'(s))^2\,dtds\right)^{\frac{1}{2}}.$$

However,

$$\begin{aligned} & \frac{1}{2(b-a)^2}\int_a^b\int_a^b (p(x,t)-p(x,s))^2\,dtds \\ = & \frac{1}{b-a}\int_a^b p^2(x,t)\,dt - \left(\frac{1}{b-a}\int_a^b p(x,t)\,dt\right)^2 \\ = & \frac{1}{b-a}\left[\int_a^x (t-a)^2\,dt + \int_x^b (t-a)^2\,dt\right] - \left(x - \frac{a+b}{2}\right)^2 \\ = & \frac{1}{b-a}\left[\frac{(x-a)^3+(b-x)^3}{3}\right] - \left(x - \frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12} \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2(b-a)^2}\int_a^b\int_a^b (f'(t)-f'(s))^2\,dtds \\ = & \frac{1}{b-a}\|f'\|_2^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2. \end{aligned}$$

Consequently, by (2.4) and (2.3), we deduce the first inequality in (2.1).

If $\gamma \le f'(t) \le \Gamma$ for a.e. $t\in(a,b)$, then, by the Grüss inequality, we have:

$$0 \le \frac{1}{b-a}\int_a^b (f'(t))^2\,dt - \left(\frac{1}{b-a}\int_a^b f'(t)\,dt\right)^2 \le \frac{1}{4}(\Gamma-\gamma)^2,$$

and the last inequality in (2.1) is proved. ∎

Corollary 1. *With the above assumptions, we have the mid-point inequality, from (2.1) with* $x=\frac{a+b}{2}$

$$\begin{aligned} & \left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ \le & \frac{(b-a)}{2\sqrt{3}}\left[\frac{1}{b-a}\|f'\|_2^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}} \\ & \left(\le \frac{(b-a)(\Gamma-\gamma)}{4\sqrt{3}}\quad \text{if}\ \ \gamma\le f'(t)\le\Gamma\ \text{ a.e. } t \text{ on } [a,b]\right). \end{aligned} \tag{2.5}$$

Remark 1. *Since* $L_\infty[a,b]\subset L_2[a,b]$ *(and the inclusion is strictly), then we remark that the inequality (2.1) can be applied also for the mappings* f *whose derivatives are unbounded on* (a,b)*, but* $f'\in L_2[a,b]$.

3. Applications for P.D.F.'s

Let X be a random variable having the p.d.f. $f:[a,b]\to\mathbb{R}_+$ and the *cumulative density function* $F:[a,b]\to[0,1]$, i.e.,

$$F(x)=\int_a^x f(t)\,dt,\quad x\in[a,b].$$

Then we have the following inequality.

Theorem 2. *With the above assumptions and if the p.d.f.* $f\in L_2[a,b]$*, then we have the inequality*

$$\begin{aligned} & \left|F(x)-\frac{b-E(X)}{b-a}-\frac{1}{b-a}\left(x-\frac{a+b}{2}\right)\right| \\ \le & \frac{1}{2\sqrt{3}}\left[(b-a)\|f\|_2^2-1\right]^{\frac{1}{2}} \\ & \left(\le \frac{(b-a)(M-m)}{4\sqrt{3}}\quad \text{if } m\le f\le M \text{ a.e. on } [a,b]\right) \end{aligned} \tag{3.1}$$

for all $x\in[a,b]$*, where* $E(X)$ *is the expectation of* X.

Proof. Put in (2.1) F instead of f to get

$$\left|F(x)-\frac{1}{b-a}\int_a^b F(t)\,dt-\frac{F(b)-F(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right| \tag{3.2}$$

$$\leq \frac{(b-a)}{2\sqrt{3}}\left[\frac{1}{b-a}\|f\|_2^2-\left(\frac{F(b)-F(a)}{b-a}\right)^2\right]^{\frac{1}{2}}$$

$$\left(\leq \frac{(b-a)(M-m)}{4\sqrt{3}} \quad \text{if } \ m\leq f(t)\leq M \text{ a.e. } t \text{ on } [a,b]\right).$$

As $F(a)=0,\ F(b)=1$, and

$$\int_a^b F(t)\,dt = b-E(X),$$

then, by (3.2), we easily deduce (3.1). ∎

Corollary 2. *With the above assumptions, we have:*

$$\left|\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{b-E(X)}{b-a}\right| \tag{3.3}$$

$$\leq \frac{1}{2\sqrt{3}}\left[(b-a)\|f\|_2^2-1\right]^{\frac{1}{2}}$$

$$\left(\leq \frac{(b-a)(M-m)}{4\sqrt{3}} \quad \textit{where } m\leq f\leq M \textit{ are as above}\right).$$

A Beta random variable X with parameters (p,q) has the probability density function

$$f(x;p,q)=\frac{x^{p-1}(1-x)^{q-1}}{B(p,q)},\ \ 0<x<1;$$

where

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}\,dt$$

is the *Euler Beta function.*

We know that

$$E(X)=\frac{p}{p+q}$$

and

$$\|f(\cdot;p,q)\|_2^2=\int_0^1\frac{x^{2(p-1)}(1-x)^{2(q-1)}}{B^2(p,q)}dx=\frac{B(2p-1,2q-1)}{B^2(p,q)}$$

and then, by Theorem 2, we may state the following proposition.

Proposition 1. *Let X be a Beta random variable with parameters (p,q). Then we have the inequality*

$$\left|\Pr(X\leq x)-\frac{p}{p+q}-x+\frac{1}{2}\right|\leq\frac{1}{2\sqrt{3}}\cdot\frac{\left[B(2p-1,2q-1)-B^2(p,q)\right]^{\frac{1}{2}}}{B(p,q)} \tag{3.4}$$

for all $x\in[0,1]$.

4. Applications for Special Means

Recall the following means.

(a) *The arithmetic mean*

$$A = A(a,b) := \frac{a+b}{2}, \quad a,b \geq 0;$$

(b) *The geometric mean*

$$G = G(a,b) := \sqrt{ab}, \quad a,b \geq 0;$$

(c) *The harmonic mean*

$$H = H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}, \quad a,b > 0;$$

(d) *The logarithmic mean*

$$L = L(a,b) := \begin{cases} a & \text{if} \quad a = b, \\ \\ \frac{b-a}{\ln b - \ln a} & \text{if} \quad a \neq b, \end{cases} \quad a,b > 0;$$

(e) *The identric mean*

$$I = I(a,b) := \begin{cases} a & \text{if} \quad a = b, \\ \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a \neq b, \end{cases} \quad a,b > 0;$$

(f) *The $p-$logarithmic mean*

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a \neq b, \\ \\ a & \text{if} \quad a = b, \end{cases} \quad a,b > 0;$$

where $p \in \mathbb{R}\backslash\{-1,0\}$ and $a,b > 0$.

The following simple relationships are well known in the literature

$$H \leq G \leq L \leq I \leq A \tag{4.1}$$

and

(4.2) L_p is monotonically increasing in $p \in \mathbb{R}$ with $L_0 := I$ and $L_{-1} := L$.

1. Consider the mapping $f(x) = x^p$, $p \in \mathbb{R}\diagdown\{-1,0\}$. Then

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(t)\,dt &= L_p^p, \\ \frac{f(b)-f(a)}{b-a} &= pL_{p-1}^{p-1}, \\ \frac{1}{b-a}\int_a^b |f'(t)|^2\,dt &= p^2 L_{2(p-1)}^{2(p-1)} \end{aligned}$$

and then, by (2.1), we have

$$\left|x^p - L_p^p - pL_{p-1}^{p-1}(x-A)\right| \leq \frac{(b-a)}{2\sqrt{3}}|p|\left[L_{2(p-1)}^{2(p-1)} - L_{p-1}^{2(p-1)}\right]^{\frac{1}{2}} \tag{4.3}$$

for all $x \in [a,b]$.

Choosing in (4.3), $x = A$, we obtain

$$\left|x^p - L_p^p\right| \le \frac{b-a}{2\sqrt{3}} |p| \left[L_{2(p-1)}^{2(p-1)} - L_{p-1}^{2(p-1)}\right]^{\frac{1}{2}} \tag{4.4}$$

for all $x \in [a,b]$.

2. Consider the mapping $f(x) = \frac{1}{x}$ $(x \in [a,b] \subset (0,\infty))$. Then

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(t)\,dt &= \frac{1}{L}, \\
\frac{f(b)-f(a)}{b-a} &= -\frac{1}{G^2}, \\
\frac{1}{b-a}\int_a^b \left|f'(t)\right|^2 dt &= \frac{a^2+ab+b^2}{3a^3b^3}, \\
\frac{1}{b-a}\int_a^b \left|f'(t)\right|^2 dt - \left(\frac{f(b)-f(a)}{b-a}\right)^2 &= \frac{(b-a)^2}{3a^3b^3}
\end{aligned}$$

and then, by (2.1), we get

$$\left|\frac{1}{x} - \frac{1}{L} + \frac{X-A}{G^2}\right| \le \frac{(b-a)^2}{6} \cdot \frac{1}{G^3} \tag{4.5}$$

for all $x \in [a,b]$.

If in (4.5) we choose $x = A$, we have

$$0 \le A - L \le \frac{(b-a)^2}{6} \cdot \frac{AL}{G^3}. \tag{4.6}$$

If in (4.5) we choose $x = L$, then we get

$$0 \le A - L \le \frac{(b-a)^2}{6} \cdot \frac{1}{G}. \tag{4.7}$$

Since we can determine that $\frac{AL}{G^2} \ge 1$ for $b \ge a$, then we can claim that (4.7) is a sharper bound than (4.6).

3. Finally, let us consider the mapping $f(x) = \ln x$, $(x \in [a,b] \subset (0,\infty))$. Then

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(t)\,dt &= \ln I, \\
\frac{f(b)-f(a)}{b-a} &= L^{-1}, \\
\frac{1}{b-a}\int_a^b \left|f'(t)\right|^2 dt &= \frac{1}{G^2}
\end{aligned}$$

and

$$\frac{1}{b-a}\int_a^b \left|f'(t)\right|^2 dt - \left(\frac{f(b)-f(a)}{b-a}\right)^2 = \frac{L^2-G^2}{G^2L^2}.$$

Applying (2.1), we get

$$\left|\ln x - \ln I - \frac{x-A}{L}\right| \le \frac{(b-a)\left(L^2-G^2\right)^{\frac{1}{2}}}{2\sqrt{3}GL} \tag{4.8}$$

for all $x \in [a,b]$.

If $x = A$, then, by (4.8), we obtain

$$I \leq \frac{A}{I} \leq \exp\left[\frac{(b-a)\left(L^2-G^2\right)^{\frac{1}{2}}}{2\sqrt{3}GL}\right]. \tag{4.9}$$

If in (4.8) we choose $x = I$, then we get

$$0 \leq A - I \leq \frac{(b-a)^2\left(L^2-G^2\right)^{\frac{1}{2}}}{2\sqrt{3}G}. \tag{4.10}$$

References

[1] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Inequalities for Functions and their Integrals and Derivatives,* Kluwer Academic, Dordrecht, 1994.

[2] A.M. FINK, Bounds on the derivation of a function from its averages, *Czechoslovak Math. J.,* **42** (1992), 289-310.

[3] G.A. ANASTASSIOU, Ostrowski type inequalities, *Proc. of the American Math. Soc.,* **123** (1995), No. 12, 3775-3781.

[4] G.A. ANASTASSIOU, Multivariate Ostrowski type inequalities, *Acta. Math. Hung.,* **76** (1997), No. 4, 267-278.

[5] S. S. DRAGOMIR and S. WANG, An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, *Computers Math. Applic.*, **33** (1997), No. 11, 15-20.

[6] M. MATIĆ, J. E. PEČARIĆ and N. UJEVIĆ, Improvement and further generalisation of some inequalities of Ostrowski-Grüss type, *Computers Math. Applic.*, (to appear).

[7] I. FEDOTOV and S. S. DRAGOMIR, An inequality of Ostrowski's type and its applications for Simpson's rule in numerical integration and for special means, *Math. Ineq. Appl.,* **2** (1999), 491-499.

[8] N. S. BARNETT and S. S. DRAGOMIR, An Ostrowski type inequality for mappings whose second derivatives are bounded and applications, *RGMIA Res. Rep. Coll.,* **1** (1998), No. 2, 69-70.

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