BETTER BOUNDS IN SOME OSTROWSKI-GRÜSS TYPE INEQUALITIES

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Abstract. The main aim of this note is to point out some improvements of the recent results in [1].

1. Introduction

As in [1], let $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ be two sequences of harmonic polynomials, that is, polynomials satisfying

$$P_n'(t) = P_{n-1}(t), \ P_0(t) = 1, \ \ t \in \mathbb{R}, \tag{1.1}$$

$$Q_n'(t) = Q_{n-1}(t), \ Q_0(t) = 1, \ \ t \in \mathbb{R}. \tag{1.2}$$

In [1], the authors proved the following result.

Lemma 1. *Let $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ be two harmonic polynomials. Set*

$$S_n(t,x) := \begin{cases} P_n(t), & t \in [a,x] \\ Q_n(t), & t \in (x,b]. \end{cases} \tag{1.3}$$

Then we have the equality

$$\begin{aligned} &\int_a^b f(t)\,dt \\ &= \sum_{k=1}^{n} (-1)^{k+1} \Big[Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) \\ &\quad - P_k(a) f^{(k-1)}(a) \Big] + (-1)^n \int_a^b S_n(t,x) f^{(n)}(t)\,dt, \end{aligned} \tag{1.4}$$

provided that $f : [a,b] \to \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$.

Using the following "pre-Grüss" inequality

$$|T(f,g)| \le \frac{1}{2}\sqrt{T(f,f)}\,(\Gamma - \gamma), \tag{1.5}$$

where

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x) g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$$

is the Chebychev functional and f, g are such that the previous integrals exist and $\gamma \le g(x) \le \Gamma$ for a.e. $x \in [a,b]$, the authors of [1] proved basically the

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following inequality for estimating the integral $\int_a^b f(t)\,dt$ in terms of the harmonic polynomials $\{P_n\}_{n\in\mathbb{N}}$, $\{Q_n\}_{n\in\mathbb{N}}$.

Theorem 1. *Assume that $f:[a,b]\to\mathbb{R}$ is such that $f^{(n)}$ is integrable and $\gamma_n \le f^{(n)}(t) \le \Gamma_n$ for all $t\in[a,b]$. Put*

$$U_n(x) := \frac{Q_{n+1}(b) - Q_{n+1}(x) + P_{n+1}(x) - P_{n+1}(a)}{b-a}.$$

Then for all $x\in[a,b]$, we have the inequality

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \sum_{k=1}^{n} (-1)^{k+1} \Big[Q_k(b) f^{(k-1)}(b) \right. \\ &+ (P_k(x) - Q_k(x)) f^{(k-1)}(x) - P_k(a) f^{(k-1)}(a)\Big] \\ &\left. - (-1)^n U_n(x) \Big[f^{(k-1)}(b) - f^{(k-1)}(a)\Big] \right| \\ \le\ & \frac{1}{2} K (\Gamma_n - \gamma_n)(b-a), \end{aligned} \tag{1.6}$$

where

$$K := \left\{ \frac{1}{b-a} \int_a^x P_n^2(t)\,dt + \int_x^b Q_n^2(t)\,dt - [U_n(x)]^2 \right\}^{\frac{1}{2}}.$$

A number of particular cases by choosing some appropriate harmonic polynomials have been obtained in [1] as well.

The main aim of this note is to point out a sharper bound in (1.6) in terms of the Euclidean norm of $f^{(n)}$ which is valid also for a larger class of mappings, i.e., for the mappings f for which $f^{(n)}$ is unbounded on (a,b) but $f^{(n)} \in L_2[a,b]$. Some particular cases as in [1], are also considered.

2. The Results

The following theorem holds.

Theorem 2. *Assume that the mapping $f:[a,b]\to\mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)} \in L_2[a,b]$ $(n\ge 1)$. If we denote*

$$\Big[f^{(n-1)}; a, b\Big] := \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a},$$

then we have the inequality

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \sum_{k=1}^{n} (-1)^{k+1} \Big[Q_k(b) f^{(k-1)}(b) \right. \\ &+ (P_k(x) - Q_k(x)) f^{(k-1)}(x) - P_k(a) f^{(k-1)}(a)\Big] \\ &\left. - (-1)^n [Q_{n+1}(b) - Q_{n+1}(x) + P_{n+1}(x) - P_{n+1}(a)] \Big[f^{(n-1)}; a, b\Big] \right| \\ \le\ & K(b-a) \left[\frac{1}{b-a} \left\| f^{(n)} \right\|_2^2 - \left(\Big[f^{(n)}; a, b\Big] \right)^2 \right]^{\frac{1}{2}} \\ & \left(\le \frac{1}{2} K (b-a)(\Gamma_n - \gamma_n) \quad \textit{if} \quad f^{(n)} \in L_\infty(a,b) \right) \end{aligned} \tag{2.1}$$

for all $x \in [a,b]$, where K is defined in Theorem 1 (and γ_n, Γ_n are as in the Introduction, i.e., $\gamma_n \leq f^{(n)}(t) \leq \Gamma_n$ for all $t \in [a,b]$.

Proof. Recall Korkine's identity

$$T(h,g) = \frac{1}{2(b-a)^2}\int_a^b\int_a^b (h(t)-h(s))(g(t)-g(s))\,dtds, \tag{2.2}$$

where $T(\cdot,\cdot)$ is the Chebychev functional defined in the Introduction. Using (2.2) and the identity (1.4), we may write (see also [1])

$$\begin{aligned}
&\int_a^b f(t)\,dt - \sum_{k=1}^{n}(-1)^{k+1}\Big[Q_k(b)\,f^{(k-1)}(b) \\
&\quad + (P_k(x)-Q_k(x))\,f^{(k-1)}(x) - P_k(a)\,f^{(k-1)}(a)\Big] \\
&\quad - (-1)^n\left[Q_{n+1}(b)-Q_{n+1}(x)+P_{n+1}(x)-P_{n+1}(a)\right]\left[f^{(n-1)};a,b\right] \\
&= (b-a)\,T\left(S_n(\cdot,x), f^{(n)}\right) \\
&= \frac{1}{2(b-a)}\int_a^b\int_a^b (S_n(t,x)-S_n(s,x))\left(f^{(n)}(t)-f^{(n)}(s)\right)dtds,
\end{aligned} \tag{2.3}$$

which is an identity that is interesting in itself as well.

Using the Cauchy-Buniakowski-Schwartz integral inequality for double integrals, we may write

$$\begin{aligned}
&\left|\int_a^b\int_a^b (S_n(t,x)-S_n(s,x))\left(f^{(n)}(t)-f^{(n)}(s)\right)dtds\right| \\
&\leq \left(\int_a^b\int_a^b (S_n(t,x)-S_n(s,x))^2\,dtds\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_a^b\int_a^b \left(f^{(n)}(t)-f^{(n)}(s)\right)^2 dtds\right)^{\frac{1}{2}} \\
&= \left[2(b-a)^2\,T(S_n(\cdot,x),S_n(\cdot,x))\right]^{\frac{1}{2}}\left[2(b-a)^2\,T\left(f^{(n)},f^{(n)}\right)\right]^{\frac{1}{2}} \\
&= 2(b-a)^2 K\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2 - \left(\left[f^{(n)};a,b\right]\right)^2\right]^{\frac{1}{2}}.
\end{aligned} \tag{2.4}$$

Now, taking the modulus in (2.3) and using the estimate (2.4), we may deduce the first inequality in (2.1).

If we assume that $f^{(n)} \in L_\infty[a,b]$ ($\subset L_2[a,b]$ and the inclusion is strict), then applying the Grüss inequality

$$\begin{aligned}
0 &\leq \frac{1}{b-a}\int_a^b \left|f^{(n)}(t)\right|^2 dt - \left(\frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt\right)^2 \\
&\leq \frac{1}{4}(\Gamma_n-\gamma_n)^2,
\end{aligned}$$

we deduce the last part in (2.1). ∎

We are now able to improve the Corollaries 1-3 and Theorem 2 from [1] as follows.

Corollary 1. *Under the assumptions of Theorem 2, we have*

$$\begin{aligned}
(2.5)\qquad & \left|\int_a^b f(t)\,dt - \sum_{k=1}^{n}\frac{(-1)^k}{k!}\left[(b-B)^k f^{(k-1)}(b)\right.\right. \\
& + \left.\left((x-A)^k-(x-B)^k\right) f^{(k-1)}(x) - (a-A)^k f^{(k-1)}(a)\right] \\
& -\frac{(-1)^n}{(n+1)!}\left[(b-B)^{n+1}-(x-b)^{n+1}\right. \\
& \left.\left. + (x-A)^{n+1}-(a-A)^{n+1}\right]\left[f^{(n-1)};a,b\right]\right| \\
\leq\; & (b-a)\,K_1\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2-\left(\left[f^{(n)};a,b\right]\right)^2\right]^{\frac{1}{2}},
\end{aligned}$$

where K_1 *is, as defined in* [1]

$$\begin{aligned}
K_1 \;:\; =&\ \frac{1}{n!}\left[\frac{(x-A)^{2n+1}-(a-A)^{2n+1}+(b-B)^{2n+1}-(x-B)^{2n+1}}{(2n+1)(b-a)}\right. \\
& \left.-\left(\frac{(b-B)^{n+1}-(x-B)^{n+1}+(x-a)^{n+1}-(a-A)^{n+1}}{(n+1)(b-a)}\right)^2\right]^{\frac{1}{2}}
\end{aligned}$$

and $x\in[a,b]$, $A,\ B\in\mathbb{R}$.

The proof follows from Theorem 2 with the polynomial choices of $P_n(t)=\frac{(t-A)^n}{n!}$ and $Q_n(t)=\frac{(t-B)^n}{n!}$ (see also [1, Corollary 1]).

Corollary 2. *Under the assumptions of Theorem 2, we have*

$$\begin{aligned}
(2.6)\qquad & \left|\int_a^b f(t)\,dt - \sum_{k=1}^{n}\frac{(-1)^{k+1}(b-a)^k}{k!\,(p+q)^k}\left[q^k\left(f^{(k-1)}(b)-(-1)^k f^{(k-1)}(a)\right)\right.\right. \\
& \left.+\left(\frac{p-q}{2}\right)^k\left[1-(-1)^k\right]f^{(k-1)}\left(\frac{a+b}{2}\right)\right] \\
& \left.-\frac{(-1)^n(b-a)^{n+1}(1+(-1)^n)}{(n+1)!\,(p+q)^{n+1}}\left[2^{n+1}+\left(\frac{p-q}{2}\right)^{n+1}\right]\left[f^{(n-1)};a,b\right]\right| \\
\leq\; & (b-a)\,K_2\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2-\left(\left[f^{(n)};a,b\right]\right)^2\right]^{\frac{1}{2}},
\end{aligned}$$

for $p,q\in\mathbb{R}$ $(p,q>0)$, *where*

$$K_2 := \frac{(b-a)^n}{n!\,(p+q)^n}\left[\frac{2\left(q^{2n+1}+\left(\frac{p-q}{2}\right)^{2n+1}\right)}{(p+q)(2n+1)}-2\left[1+(-1)^n\right]\frac{\left(q^{n+1}+\left(\frac{p-q}{2}\right)^{n+1}\right)^2}{(n+1)^2(p+q)^2}\right]^{\frac{1}{2}}.$$

The proof follows by Corollary 1 with $A=\frac{pa+qb}{p+q}$, $x=\frac{a+b}{2}$ and $B=\frac{qa+pb}{p+q}$ where $p,q\in\mathbb{R}$ and $p+q>0$ (see also [1, Corollary 2]).

For $x=b$, Theorem 2 gives the following.

Theorem 3. *With the assumptions in Theorem 2, we have:*

$$\begin{aligned}(2.7)\qquad & \left|\int_a^b f(t)\,dt - \sum_{k=1}^{n}(-1)^{k+1}\left[P_k(b)\,f^{(k-1)}(b) - P_k(a)\,f^{(k-1)}(a)\right]\right.\\ & \left. -(-1)^n\left[P_{n+1}(b)-P_{n+1}(a)\right]\left[f^{(n-1)};a,b\right]\right| \\ \leq\ & K_3\,(b-a)\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2 - \left(\left[f^{(n)};a,b\right]\right)^2\right]^{\frac{1}{2}},\end{aligned}$$

where K_3 *is given by (see* [1, Theorem 2]*)*

$$K_3 := \left[\frac{1}{b-a}\int_a^x P_n^2(t)\,dt - \left(\frac{P_{n+1}(b)-P_{n+1}(a)}{b-a}\right)^2\right]^{\frac{1}{2}}.$$

The choice $P_n(t) = \frac{1}{n!}\left(t - \frac{a+b}{2}\right)^n$ provides the following corollary.

Corollary 3. *Under the assumptions of Theorem 2, we have:*

$$\begin{aligned}(2.8)\qquad & \left|\int_a^b f(t)\,dt - \sum_{k=1}^{n}\frac{(-1)^{k+1}}{2^k k!}(b-a)^k\left[f^{(k-1)}(b) - (-1)^k f^{(k-1)}(a)\right]\right.\\ & \left. -\frac{(-1)^n\left(1+(-1)^n\right)}{2^{n+1}(n+1)!}(b-a)^{n+1}\left[f^{(n-1)};a,b\right]\right| \\ \leq\ & K_4\,(b-a)\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2 - \left(\left[f^{(n)};a,b\right]\right)^2\right]^{\frac{1}{2}},\end{aligned}$$

where K_4 *is given by (see* [1, Corollary 3]*)*

$$K_4 := \frac{(b-a)^n}{n!2^n}\left[\frac{1}{2n+1} - \frac{\left(1+(-1)^n\right)^2}{(n+1)^2}\right]^{\frac{1}{2}}.$$

Remark 1. *All the other results from Sections 4 and 5 can be improved accordingly. For example, if we assume that the derivative* $f^{(n)} \in L_2[a,b]$ $(n \in \{1,2,3,4\})$*, then we have the Simpson's inequality (for* $n \in \{1,2,3\}$*)*

$$\begin{aligned}(2.9)\qquad & \left|\int_a^b f(t)\,dt - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\ \leq\ & \tilde{c}_n\,(b-a)^n\,\sigma\left(f^{(n)};a,b\right)\end{aligned}$$

where

$$\tilde{c}_1 = \frac{1}{6},\ \tilde{c}_2 = \frac{1}{12\sqrt{30}},\ \tilde{c}_3 = \frac{1}{48\sqrt{105}}$$

and

$$\sigma\left(f^{(n)};a,b\right) := \left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2 - \left(\left[f^{(n)};a,b\right]\right)^2\right]^{\frac{1}{2}},\ \ n \in \{1,2,3,4\}.$$

For $n = 4$, we have the perturbed Simpson's inequality:

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] + \frac{(b-a)^5}{2880}\left[f^{(3)}; a, b \right] \right| \leq \frac{1}{2880}\sqrt{\frac{11}{14}}\,(b-a)^4\,\sigma\left(f^{(4)}; a, b \right). \tag{2.10}$$

References

[1] C.E.M. PEARCE, J.E. PEČARIĆ, N. UJEVIĆ and S. VARONŠANEC, Generalisations of some inequalities of Ostrowski-Grüss type, *Math. Ineq. & Appl.,* **3** (1) (2000), 25-34.

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