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CAUCHY–SCHWARZ FUNCTIONALS

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Abstract. We treat a class of functionals which satisfy the Cauchy–Schwarz inequality. This appears to be a natural unifying concept and subsumes *inter alia* isotonic linear functionals and sublinear positive isotonic functionals. Striking superadditivity and monotonicity properties are derived.

1. Introduction

One of the oldest classical inequalities is that associated with the names Cauchy, Buniakowski and Schwarz. This inequality, which for brevity we term the Cauchy–Schwarz inequality, states in its discrete form that if $a_i, b_i \in R$ $(i = 1, 2, ..., n)$, then

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \geq \left(\sum_{i=1}^{n} a_i b_i \right)^2 .$$

Equality holds if and only if $a_i = rb_i$ for all $i = 1, 2, ..., n$ and $r \in R$.

Various proofs of this inequality, as well as results connected with it, are given in the book of Mitrinović, Pečarić and Fink [10, Chapter 4] along with further references. Despite its antiquity, this result admits numerous recent developments in general settings (see, for example [1–9]).

Key words and Phrases : Cauchy–Schwarz inequality, sublinear functionals, superadditivity, monotonicity, isotonic linear functionals

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In integral form, the Cauchy–Schwarz inequality reads

$$\int_a^b f^2(x)dx \int_a^b g^2(x)dx \geq \left(\int_a^b f(x)g(x)dx\right)^2,$$

where $f, g : [a, b] \to R$ are Riemann–integrable functions.

Let E be a nonempty set and L a class of real–valued functions on E possessing the properties

(L_1) $f, g \in L \Rightarrow af + bg \in L$ for all $a, b \in R$;

(L_2) $1 \in L$, that is, if $f(t) = 1$ for all $t \in E$, then $f \in L$.

A functional $A : L \to R$ is termed a *positive linear functional* if the conditions

(A_1) $A(af + bg) = aA(f) + bA(g)$ for $f, g \in L$ and $a, b \in R$;

(A_2) $f \in L$ and $f(t) \geq 0$ on E imply $A(f) \geq 0$

are satisfied.

If $w \geq 0$ and $wf^2, wg^2, wfg \in L$, then the Cauchy–Schwarz inequality

$$A(wf^2)A(wg^2) \geq |A(wfg)|^2$$

holds for each positive linear functional A on L.

We are now ready for an overview of the paper.

In Section 2 we introduce a natural class K of real–valued functions on a nonempty set E and define the Cauchy–Schwarz class $CS(K, R)$ of functionals on K, also in a natural way. It is known that isotonic linear functionals on K belong to $CS(K, R)$. We show that sublinear positive functionals do also, as well as a further class of sublinear functionals that we term *solid*. We conclude Section 2 by proving that $CS(K, R)$ is a convex cone in the linear space of real–valued mappings on K.

In Sections 3 and 4 we establish striking superadditivity and monotonicity properties of functionals related intrinsically to the class $CS(K, R)$. Section 5 provides a strengthening of the results of Section 4 in a particular case. In Section 6 we conclude by remarking on a few basic examples.

2. Cauchy–Schwarz functionals

Suppose E is a nonempty set and $K = K(E)$ a class of real–valued functions on E with the properties

(K_1) $f, g \in K \;\Rightarrow\; f + g \in K$;
(K_2) $f \in K, \alpha \geq 0 \;\Rightarrow\; \alpha f \in K$;
(K_3) $f, g \in K \;\Rightarrow\; fg \in K$;
(K_4) $f \in K \;\Rightarrow\; |f| \in K$.

Definition 2.1. We say that a real–valued functional $A : K \to R$ is of *Cauchy–Schwarz type* on K (written $A \in CS(K, R)$) if

$$A(f^2)A(g^2) \geq [A(fg)]^2 \quad \text{for all} \quad f, g \in K.$$

Definition 2.2. An *isotonic linear functional* $A : K \to R$ is a mapping satisfying

(I_1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in K$ and $\alpha, \beta \in R$;
(I_2) $f \in K$ and $f \geq 0$ (that is, $f(t) \geq 0$ for all $t \in E$) $\Rightarrow$ $A(f) \geq 0$.

It is well–known that such an A satisfies $A \in CS(K, R)$ (see [15, p. 135]).

Definition 2.3. A functional $A : K \to R$ is *sublinear* and *positive isotonic* when

(S_1) $A(f + g) \leq A(f) + A(g)$ for all $f, g \in K$;
(S_2) $A(\alpha f) = \alpha A(f)$ for all $\alpha \geq 0$ and $f \in K$;
(S_3) If $0 \leq f \leq g$, then $A(f) \leq A(g)$;
(S_4) $|A(f)| \leq A(|f|)$ for all $f \in K$.

We now give our first result.

Theorem 2.4. *Every sublinear and positive isotonic functional on K belongs to the class $CS(K, R)$.*

Proof. Suppose A is sublinear and positive isotonic. For every $t, z \in E$ and $f, g \in K(E)$, we have by the Cauchy–Schwarz inequality for real numbers that

$$f^2(t)g^2(z) + f^2(z)g^2(t) \geq 2|f(t)g(t)||f(z)g(z)|,$$

so that

$$f^2(t)g^2 + g^2(t)f^2 \geq 2|f(t)g(t)||fg| \tag{2.1}$$

for all $t \in E$. Applying the functional A to this inequality yields

$$\begin{aligned} f^2(t)A(g^2) + g^2(t)A(f^2) &\geq A[f^2(t)g^2 + g^2(t)f^2] \quad \textit{by } (S_1) \\ &\geq A[2|f(t)g(t)|\ |fg|] \quad \textit{by } (2.1) \textit{ and } (S_3) \\ &= 2|f(t)g(t)|A(|fg|) \quad \textit{by } (S_2) \end{aligned}$$

for all $t \in E$. Hence

$$A(g^2)f^2 + A(f^2)g^2 \geq 2A(|fg|)\ |fg|.$$

Applying the functional A again provides

$$\begin{aligned} 2A(f^2)A(g^2) &\geq A[A(g^2)f^2 + A(f^2)g^2] \quad \textit{by } (S_1) \\ &\geq A[2A(|fg|)\ |fg|] \quad \textit{by } (2.2) \textit{ and } (S_3) \\ &= 2[A(|fg|)]^2 \quad \textit{by } (S_2). \end{aligned}$$

Thus by (S_4) we have proved in particular that

$$A(f^2)A(g^2) \geq [A(|fg|)]^2$$

as required.

Definition 2.5 A functional $A : K \to R_+$ is said to be *sublinear* and *solid* if

(0_1) $A(f + g) \leq A(f) + A(g)$ for all $f, g \in K$;
(0_2) $A(\alpha f) = \alpha A(f)$ for all $\alpha \geq 0$ and $f \in K$;
(0_3) $|f| \leq |g| \ \Rightarrow\ A(f) \leq A(g)$.

The following theorem also holds.

Theorem 2.6. *Every sublinear and solid functional on K belongs to the class $CS(K,R)$.*

Proof. Conditions (O_1) and (O_2) are the same as (S_1), (S_2), while (O_3) matches (S_3) for $f,g \geq 0$. As (S_4) is used only in the last step in the proof of the previous theorem, we have by the argument in that proof that

$$A(f^2)A(g^2) \geq [A(|fg|)]^2. \tag{2.3}$$

Now $|\,|f|\,| = |f|$, so by (O_3) we have both $A(|f|) \leq A(f)$ and $A(f) \leq A(|f|)$ and thus $A(|f|) = A(f)$ for all $f \in K$. Hence

$$A(f^2)A(g^2) \geq [A(fg)]^2.$$

by (2.3).

Remark 2.7. From the proofs, we have that sublinear and positive isotonic functionals and sublinear and solid functionals both in fact satisfy (2.3).

We now address the structure of $CS(K,R)$.

Theorem 2.8. *The set $CS(K,R)$ is a convex cone in the linear space of all real–valued mappings on K, that is,*

(C_1) $A, B \in CS(K,R) \ \Rightarrow \ A + B \in CS(K,R)$;
(C_2) $A \in CS(K,R)$ and $\alpha \geq 0 \ \Rightarrow \ \alpha A \in CS(K,R)$.

Proof. Suppose $A, B \in CS(K,R)$. Then

$$[A(f^2)]^{1/2}[A(g^2)]^{1/2} \geq |A(fg)| \quad \textit{and} \quad [B(f^2)]^{1/2}[B(g^2)]^{1/2} \geq |B(fg)|$$

for all $f, g \in K$, which give on addition that

$$\begin{aligned} [A(f^2)]^{1/2}[A(g^2)]^{1/2} + [B(f^2)]^{1/2}[B(g^2)]^{1/2} &\geq |A(fg)| + |B(fg)| \\ &\geq |(A+B)(fg)| \end{aligned}$$

for all $f, g \in K$. On the other hand, from the elementary inequality

$$(a^2 + b^2)^{1/2}(c^2 + d^2)^{1/2} \geq ac + bd$$

for $a, b, c, d \geq 0$,

$$\begin{aligned}
&[A(f^2)]^{1/2}[A(g^2)]^{1/2} + [B(f^2)]^{1/2}[B(g^2)]^{1/2} \\
&\quad \leq \{A(f^2) + B(f^2)\}^{1/2}\{A(g^2) + B(g^2)\}^{1/2} \\
&\quad = [(A + B)(f^2)]^{1/2}[(A + B)(g^2)]^{1/2},
\end{aligned}$$

so that

$$[(A + B)(f^2)][(A + B)(g^2)] \geq |(A + B)(fg)|^2$$

for all $f, g \in K$, that is, $A + B \in CS(K, R)$.

The second condition is clear.

3. Superadditivity and monotonicity of μ

Consider the functional $\mu : CS(K, R) \times K^2 \to R$ given by

$$\mu(A, f, g) := [A(f^2)]^{1/2}[A(g^2)]^{1/2} - |A(fg)|.$$

We can verify immediately the following properties for all $A \in CS(K, R)$ and $f, g \in K$.

(*i*) $\mu(A, f, g) \geq 0$;

(*ii*) $\mu(A, f, g) = \mu(A, g, f)$;

(*iii*) $\mu(\alpha A, f, g) = \alpha\mu(A, f, g)$ for all $\alpha \geq 0$.

Further, we have the following result for the mapping $\mu(\cdot, f, g)$.

Theorem 3.1.

(i) μ *is superadditive*;

(ii) μ *is monotone nondecreasing.*

Proof. (i) We have for $A, B \in CS(K, R)$ that

$$\begin{aligned}\mu(A+B,f,g)&\\ &= [A(f^2)] + B(f^2)]^{1/2}[A(g^2) + B(g^2)]^{1/2} - |A(fg) + B(fg)|\\ &\geq [A(f^2)]^{1/2}[A(g^2)]^{1/2} + [B(f^2)]^{1/2}[B(g^2)]^{1/2} - |A(fg)| - |B(fg)|\\ &= \mu(A,f,g) + \mu(B,f,g).\end{aligned}$$

(ii) Suppose $A, B \in CS(K, R)$ with $A \geq B$, that is, $A - B \in CS(K, R)$. Then

$$\mu(A, f, g) = \mu((A - B) + B, f, g) \geq \mu(A - B, f, g) + \mu(B, f, g).$$

Since μ is nonnegative, we have

$$\mu(A, f, g) \geq \mu(B, f, g),$$

completing the proof.

Now, suppose that $\mathcal{A}(E)$ is a nonempty family of subsets of E satisfying
(P_1) $\quad I, J \in \mathcal{A}(E) \;\Rightarrow\; I \cup J \in \mathcal{A}(E)$;
(P_2) $\quad I, J \in \mathcal{A}(E) \;\Rightarrow\; I \setminus J \in \mathcal{A}(E)$.

We represent by $\chi_I : E \to \{0, 1\}$ the characteristic mapping of I, that is,

$$\chi_I(t) = \begin{cases} 1 & \text{if } t \in I \\ 0 & \text{if } t \in E \setminus I. \end{cases}$$

Definition 3.2. A class of functions K defined on E is a *hereditary class related to the family* $\mathcal{A}(E)$ if
(H) $\quad f \in K$ implies that $\chi_I \cdot f \in K$ for all $I \in \mathcal{A}(E)$.

For such a class K, we introduce the mapping $\eta : \mathcal{A}(E) \times CS(K, R) \times K^2 \to R$, defined by

$$\eta(I, A, f, g) := [A(\chi_I f^2)]^{1/2}[A(\chi_I g^2)]^{1/2} - |A(\chi_I fg)|.$$

Remark 3.3. For every fixed $I \in \mathcal{A}(E)$, the mapping $\eta(I, \cdot, f, g)$ is superadditive and monotone nondecreasing on $CS(K, R)$. This follows by an argument similar to that in the proof of the preceding theorem.

We now consider the properties of η as a function defined on $\mathcal{A}(E)$.

Theorem 3.4. *Let K be a hereditary class of functions related to the family $\mathcal{A}(E)$. If A is an isotonic linear functional on K and $f, g \in K$, then the following hold:*
(i) *$\eta(\cdot, A, f, g)$ is superadditive on $\mathcal{A}(E)$;*
(ii) *$\eta(\cdot, A, f, g)$ is monotone nondecreasing on $\mathcal{A}(E)$.*

Proof. (i) Suppose $I, J \in \mathcal{A}(E)$ with $I \cap J = \emptyset$. Then

$$\begin{aligned}\eta(I \cup J, A, f, g) &= [A(\chi_I f^2) + A(\chi_J f^2)]^{1/2}[A(\chi_I g^2) + A(\chi_J g^2)]^{1/2} \\ &\qquad - |A(\chi_I fg) + A(\chi_J fg)| \\ &\geq [A(\chi_I f^2)]^{1/2}[A(\chi_I g^2)]^{1/2} + [A(\chi_J f^2)]^{1/2}[A(\chi_J g^2)]^{1/2} \\ &\qquad - |A(\chi_I fg)| - |A(\chi_J fg)| \\ &= \eta(I, A, f, g) + \eta(J, A, f, g).\end{aligned}$$

(ii) Suppose $I, J \in \mathcal{A}(E)$ with $J \subseteq I$. Then by part (i)

$$\eta(I, A, f, g) = \eta((I \setminus J) \cup J, A, f, g) \geq \eta(I \setminus J, A, f, g) + \eta(J, A, f, g).$$

Since η is nonnegative, it follows that

$$\eta(I, A, f, g) \geq \eta(J, A, f, g),$$

and we are done.

Corollary 3.5. If $\phi(\cdot)$ is monotone nondecreasing and superadditive, then $\phi(\mu)$ inherits the properties of μ in Theorem 3.1 and $\phi(\eta)$ those of η in Remark 3.3 and Theorem 3.4.

4. Superadditivity and monotonicity of γ

Suppose that K is a hereditary class related to $\mathcal{A}(E)$ and consider the mapping $\gamma : \mathcal{A}(E) \times CS(K, R) \times K^2 \to R$ given by

$$\gamma(I, A, f, g) := (A(\chi_I f^2)A(\chi_I g^2) - [A(\chi_I fg)]^2)^{1/2}.$$

It is evident that for all $A \in CS(K, R)$, $I \in \mathcal{A}(E)$ and $f, g \in K$, we have
(i) $\gamma(I, A, f, g) \geq 0$;
(ii) $\gamma(I, A, f, g) = \gamma(I, A, g, f)$;
(iii) $\gamma(I, k, f, g) = k\gamma(I, A, f, g)$ for all $k \geq 0$.

An important property of this functional is given by the following theorem.

Theorem 4.1. *The mapping $\gamma(I, \cdot, f, g)$ is superadditive on $CS(K, R)$.*

Proof. Suppose $A, B \in CS(K, R)$. We have

$$\begin{aligned}\gamma^2(I, A+B, f, g) &= [A(\chi_I f^2)] + B(\chi_I f^2)][A(\chi_I g^2) + B(\chi_I g^2)] \\ &\quad - ([A(\chi_I fg)]^2 + 2A(\chi_I fg)B(\chi_I fg) + [B(\chi_I fg)]^2) \\ &= \gamma^2(I, A, f, g) + \gamma^2(I, B, f, g) + A(\chi_I f^2)B(\chi_I g^2) \\ &\quad + B(\chi_I f^2)A(\chi_I g^2) - 2A(\chi_I fg)B(\chi_I fg). \qquad (4.1)\end{aligned}$$

We now prove that

$$\begin{aligned}&A(\chi_I f^2)B(\chi_I g^2) + B(\chi_I f^2)A(\chi_I g^2) - 2A(\chi_I fg)B(\chi_I fg) \\ &\quad \geq 2\gamma(I, A, f, g)\gamma(I, B, f, g). \qquad (4.2)\end{aligned}$$

Set

$$\begin{aligned}a &= [A(\chi_I f^2)]^{1/2}, \quad b = [A(\chi_I g^2)]^{1/2}, \quad x = A(\chi_I fg), \\ c &= [B(\chi_I f^2)]^{1/2}, \quad d = [B(\chi_I g^2)]^{1/2}, \quad y = B(\chi_I fg).\end{aligned}$$

By the definition and nonnegativity of γ, we have

$$ab - x > 0 \quad \textit{and} \quad dc > y. \qquad (4.3)$$

We have to prove that

$$a^2d^2 + b^2c^2 - 2xy \geq 2(a^2b^2 - x^2)^{1/2}(d^2c^2 - y^2)^{1/2}.$$

By (4.3), both sides are nonnegative, so our task is to establish

$$(a^2d^2 + b^2c^2 - 2xy)^2 \geq 4(a^2b^2 - x^2)(d^2c^2 - y^2).$$

By a simple calculation,

$$(abcd - xy)^2 \geq (a^2b^2 - x^2)(d^2c^2 - y^2)$$

so it suffices to show that

$$(a^2d^2 + b^2c^2 - 2xy)^2 \geq 4(abcd - xy)^2$$

or, since again both expressions in parentheses are nonnegative, that

$$a^2d^2 + b^2c^2 - 2xy \geq 2(abcd - xy),$$

which is immediate.

We have from (4.1) and (4.2) that

$$\begin{aligned}\gamma^2(I, A + B, f, g) &\geq \gamma^2(I, A, f, g) + \gamma^2(I, B, f, g) + 2\gamma(I, A, f, g)\gamma(I, B, f, g) \\ &= (\gamma(I, A, f, g) + \gamma(I, B, f, g))^2, \qquad (4.4)\end{aligned}$$

and so by the nonnegativity of γ

$$\gamma(I, A + B, f, g) \geq \gamma(I, A, g, f) + \gamma(I, B, f, g),$$

as required.

Remark 4.2. The class K is trivially a hereditary class related to $\mathcal{A}(E) = \{E, \emptyset\}$. Thus the map $\gamma_0(A, f, g) := \gamma(E, A, f, g)$, which is given by

$$\gamma_0(A, f, g) := (A(f^2)A(g^2) - [A(fg)]^2)^{1/2},$$

is superadditive on $CS(K, R)$.

Theorem 4.3. *Let A be an isotonic linear functional on K. Then the mapping $\gamma(\cdot, A, f, g)$ is superadditive as an index–set mapping on $\mathcal{A}(E)$.*

Proof. Suppose $I, J \in \mathcal{A}(E)$ with $I \cap J = \emptyset$. Then

$$\begin{aligned}\gamma^2(I \cup J, A, f, g) &= (A(\chi_I f^2) + A(\chi_J f^2))(A(\chi_I g^2) + A(\chi_J g^2)) \\ &\quad - (A(\chi_I fg) + A(\chi_J fg))^2 \\ &= \gamma^2(I, A, f, g) + \gamma^2(J, A, f, g) + A(\chi_I f^2)A(\chi_J g^2) \\ &\quad + A(\chi_J f^2)A(\chi_I g^2) - 2A(\chi_I fg)A(\chi_J fg).\end{aligned}$$

Arguing as in the previous theorem, we have

$$\begin{aligned}&A(\chi_I f^2)A(\chi_J g^2) + A(\chi_J f^2)A(\chi_I g^2) - 2A(\chi_I fg)A(\chi_J fg) \\ &\quad \geq 2\gamma(I, A, f, g)\gamma(J, A, f, g), \qquad (4.5)\end{aligned}$$

so that

$$\gamma(I \cup J, A, f, g) \geq \gamma(I, A, f, g) + \gamma(J, A, f, g)$$

and the proof is complete.

Corollary 4.4. If $\phi(\cdot)$ is monotone nondecreasing and superadditive, then $\phi(\gamma)$ inherits the properties of γ in Theorems 4.1 and 4.3.

Remark 4.5. We have from Corollary 4.4 or (4.4) that

$$\beta(I,A,f,g) := \gamma^2(I,A,f,g)$$

is superadditive on $CS(K,R)$. However stronger results exist, as we shall see in the next section.

5. Strong superadditivity and monotonicity of β

In this section, we study the nonnegative functional β introduced in the preceding section, and given by

$$\beta(I,A,f,g) := A(\chi_I f^2)A(\chi_I g^2) - [A(\chi_I fg)]^2.$$

Theorem 5.1. *The following hold:*

(i) $\beta(\cdot,f,g)$ *is strongly superadditive on* $CS(K,R)$*, that is, if* $A,B \in CS(K,R)$*, then*

$$\beta(I,A+B,f,g) - \beta(I,A,f,g) - \beta(I,B,f,g) \geq \left(\det\begin{vmatrix} [A(\chi_I f^2)]^{1/2} & [A(\chi_I g^2)]^{1/2} \\ [B(\chi_I f^2)]^{1/2} & [B(\chi_I g^2)]^{1/2} \end{vmatrix}\right)^2 \geq 0;$$

(ii) $\beta(\cdot,f,g)$ *is strongly monotone nondecreasing on* $CS(K,R)$*, that is, if* $A \geq B$*, then*

$$\beta(I,A,f,g) - \beta(I,B,f,g) \geq \left(\det\begin{vmatrix} [A(\chi_I f^2) - B(\chi_I f^2)]^{1/2} & [A(\chi_I g^2) - B(\chi_I g^2)]^{1/2} \\ [B(\chi_I f^2)]^{1/2} & [B(\chi_I g^2)]^{1/2} \end{vmatrix}\right)^2 \geq 0.$$

Proof. (i) Suppose $A,B \in C(S,K)$. We have from (4.1) that

$$\beta(I,A+B,f,g) = \beta(I,A,f,g) + \beta(I,B,f,g) + A(\chi_I f^2)B(\chi_I g^2) + B(\chi_I f^2)A(\chi_I g^2) - 2A(\chi_I fg)B(\chi_I fg). \tag{5.1}$$

Since $A, B \in CS(K, R)$,

$$|A(\chi_I fg)| \le [A(\chi_I f^2)A(\chi_I g^2)]^{1/2}, \quad |B(\chi_I fg)| \le [B(\chi_I f^2)B(\chi_I g^2)]^{1/2}$$

and thus

$$\begin{aligned} A(\chi_I fg)B(\chi_I fg) &\le |A(\chi_I fg)B(\chi_I fg)| \\ &\le [A(\chi_I f^2)B(\chi_I g^2)]^{1/2}[A(\chi_I g^2)B(\chi_I f^2)]^{1/2}. \end{aligned}$$

The desired result is immediate from this result and (5.1).

(ii) If $A \ge B$, we have

$$\begin{aligned} &\beta(I, A, f, g) - \beta(I, A - B, f, g) - \beta(I, B, f, g) \\ &\quad \ge \left(\det \begin{vmatrix} [A(\chi_I f^2) - B(\chi_I f^2)]^{1/2} & [A(\chi_I g^2) - B(\chi_I g^2)]^{1/2} \\ [B(\chi_I f^2)]^{1/2} & [B(\chi_I g^2)]^{1/2} \end{vmatrix}\right)^2 \ge 0 \end{aligned}$$

and we are done.

Theorem 5.2. *Suppose A is an isotonic linear functional on K. We have the following.*

(i) *$\beta(\cdot, A, f, g)$ is strongly superadditive on $\mathcal{A}(E)$, that is, if $I \cap J = \emptyset$, then*

$$\begin{aligned} &\beta(I \cup J, A, f, g) - \beta(I, A, f, g) - \beta(J, A, f, g) \\ &\quad \ge \left(\det \begin{vmatrix} [A(\chi_I f^2)]^{1/2} & [A(\chi_I g^2)]^{1/2} \\ [A(\chi_J f^2)]^{1/2} & [A(\chi_J g^2)]^{1/2} \end{vmatrix}\right)^2 \ge 0. \end{aligned}$$

(ii) *$\beta(\cdot, A, f, g)$ is strongly monotone nondecreasing on $\mathcal{A}(E)$, that is, if $I, J \in \mathcal{A}(E)$ and $J \subseteq I$, then*

$$\begin{aligned} &\beta(I, A, f, g) - \beta(J, A, f, g) \\ &\quad \ge \left(\det \begin{vmatrix} [A(\chi_I f^2) - A(\chi_J f^2)]^{1/2} & [A(\chi_I g^2) - A(\chi_J g^2)]^{1/2} \\ [A(\chi_J f^2)]^{1/2} & [A(\chi_J g^2)]^{1/2} \end{vmatrix}\right)^2. \end{aligned}$$

Proof. (i) Let $I, J \in \mathcal{A}(E)$ with $I \cap J = \emptyset$. Then

$$\begin{aligned} \beta(I \cup J, A, f, g) &= (A(\chi_I f^2) + A(\chi_J f^2))(A(\chi_I g^2) + A(\chi_J g^2)) \\ &\quad - (A(\chi_I fg) + A(\chi_J fg))^2 \\ &= \beta(I, A, f, g) + \beta(J, A, f, g) + A(\chi_I f^2)A(\chi_J g^2) \\ &\quad + A(\chi_J f^2)A(\chi_I g^2) - 2A(\chi_I fg)A(\chi_J fg). \end{aligned}$$

As in Theorem 5.1, we deduce the inequality of part (i), which implies in turn that of part (ii).

6. Applications

In this short section we note some immediate applications. First we define the classes of sequences

$$\begin{aligned} J &= \{a = (a_n)_{n\in N} : a_n \in R \text{ for all } n \in N\},\\ P &= \{I \subset N : I \text{ is finite}\},\\ J_+ &= \{p = (p_n)_{n\in N} : p_n \geq 0 \text{ for all } n \in N\}. \end{aligned}$$

Consider the functional $\mu : P \times J_+ \times J^2 \to R$ given by

$$\mu(I, p, a, b) := \left[\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2\right]^{1/2} - \left|\sum_{i\in I} p_i a_i b_i\right|.$$

We have

$$\mu(I, p, a, b) = \mu(A_{I,p}, a, b),$$

where $A_{I,p}(x) = \sum_{i\in I} p_i x_i$ is an isotonic linear functional which belongs to $CS(J, R)$. Theorems 3.1 and 3.4 apply to μ.

Similarly Theorems 4.1 and 4.3 apply to the mapping $\gamma : P \times J_+ \times J^2 \to R$ given by

$$\gamma(I, p, a, b) := \left[\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 - \left(\sum_{i\in I} p_i a_i b_i\right)^2\right]^{1/2}$$

and Theorems 5.1 and 5.2 to the mapping $\beta : P \times J_+ \times J^2 \to R$ given by

$$\beta(I, p, a, b) := \sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 - \left(\sum_{i\in I} p_i a_i b_i\right)^2,$$

Similar applications hold for Riemann–integrable functions. Let $[a, b]$ be a real interval and denote by $R[a, b]$ the algebra of Riemann–integrable functions on $[a, b]$ and by $R_+[a, b]$ the class of nonnegative functions belonging to $R[a, b]$. Define the functional $\mu : R_+[a, b] \times R^2[a, b] \to R$ by

$$\mu(a, b; h, f, g) := \left[\int_a^b h(x)f^2(x)dx) \int_a^b h(x)g^2(x)dx\right]^{1/2} - \left|\int_a^b h(x)f(x)g(x)dx\right|.$$

Then

$$\mu(a, b; h, f, g) = \mu(A_{[a,b],h}, f, g),$$

where $A_{[a,b],h}(f) = \int_a^b h(x)f(x)dx$, is an isotonic linear functional which belongs to $CS(R[a, b], R)$. Clearly Theorems 3.1 and 3.4 apply to μ.

Similarly Theorems 4.1 and 4.3 apply to the mapping $\gamma : R_+[a, b] \times R[a, b] \to R$ given by

$$\gamma(a, b; h, f, g) := \left[\int_a^b h(x)f^2(x)dx) \int_a^b h(x)g^2(x)dx] - \left(\int_a^b h(x)f(x)g(x)dx\right)^2\right]^{1/2}$$

and Theorems 5.1 and 5.2 to the mapping $\beta : R_+[a, b] \times R^2[a, b] \to R$ given by

$$\beta(a, b; h, f, g) := \int_a^b h(x)f^2(x)dx \int_a^b h(x)g^2(x)dx - \left(\int_a^b h(x)f(x)g(x)dx\right)^2.$$

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