

# ON ACZÉL'S INEQUALITY FOR REAL NUMBERS

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ABSTRACT. In this note, we point out some new inequalities of Aczel's type for real numbers.

## I. Introduction

In 1956, J. Aczél has proved the following interesting inequality ([2, p. 57], [3, p. 117 ]):

**Theorem A.** *Let  $a = (a_1, a_2, \dots, a_n)$  and  $b = (b_1, b_2, \dots, b_n)$  be two sequences of real numbers such that*

$$0 < a_1^2 - a_2^2 - \dots - a_n^2 \quad \text{or} \quad 0 < b_1^2 - b_2^2 - \dots - b_n^2.$$

*Then*

$$(1.1) \quad \begin{aligned} & (a_1^2 - a_2^2 - \dots - a_n^2)(b_1^2 - b_2^2 - \dots - b_n^2) \\ & \leq (a_1 b_1 - a_2 b_2 - \dots - a_n b_n)^2, \end{aligned}$$

*with the equality if and only if the sequences  $a$  and  $b$  are proportional.*

For various generalizations of Theorem A, see the recent book ([3, p. 117]) where further references are given.

Now, in this note, we give another proof than that embodied in [2, p. 57] for a weighted variant of (1.1).

Assume that

$$\sum_{i=1}^n p_i a_i^2 \leq a^2 \quad \text{and} \quad \sum_{i=1}^n p_i b_i^2 \leq b^2,$$

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where  $a_i, b_i, a, b \in R$  and  $0 \leq p_i$  for  $i = 1, 2, \dots, n$ . Then we have the following inequality:

$$(1.2) \quad \left(a^2 - \sum_{i=1}^n p_i a_i^2\right) \left(b^2 - \sum_{i=1}^n p_i b_i^2\right) \leq \left(ab - \sum_{i=1}^n p_i a_i b_i\right)^2.$$

Indeed, by a simple calculation, we have

$$(a^2 - c^2)(b^2 - d^2) \leq (|ab| - |cd|)^2$$

for all  $a, b, c, d \in R$ . Thus we have

$$(1.3) \quad \begin{aligned} & \left(a^2 - \sum_{i=1}^n p_i a_i^2\right) \left(b^2 - \sum_{i=1}^n p_i b_i^2\right) \\ & \leq \left(|ab| - \left(\sum_{i=1}^n p_i a_i^2\right)^{1/2} \left(\sum_{i=1}^n p_i b_i^2\right)^{1/2}\right)^2. \end{aligned}$$

By Cauchy-Buniakowski-Schwarz's inequality, we have

$$\left|\sum_{i=1}^n p_i a_i b_i\right| \leq \left(\sum_{i=1}^n p_i a_i^2\right)^{1/2} \left(\sum_{i=1}^n p_i b_i^2\right)^{1/2}$$

and so

$$\begin{aligned} 0 & \leq |ab| - \left(\sum_{i=1}^n p_i a_i^2\right)^{1/2} \left(\sum_{i=1}^n p_i b_i^2\right)^{1/2} \\ & \leq |ab| - \left|\sum_{i=1}^n p_i a_i b_i\right| \\ & = \left||ab| - \left|\sum_{i=1}^n p_i a_i b_i\right|\right| \\ & \leq \left|ab - \sum_{i=1}^n p_i a_i b_i\right|. \end{aligned}$$

Thus, we have

$$(1.4) \quad \left(|ab| - \left(\sum_{i=1}^n p_i a_i^2\right)^{1/2} \left(\sum_{i=1}^n p_i b_i^2\right)^{1/2}\right)^2 \leq \left(ab - \sum_{i=1}^n p_i a_i b_i\right)^2.$$

Therefore, from (1.3) and (1.4), we have the inequality (1.2). This completes the proof.

## II. The Results

We will start with the following theorem which give a refinement of the following variant of Aczél's inequality:

$$(2.1) \quad \left(a^2 - \sum_{i=1}^n p_i a_i^2\right)^{1/2} \left(b^2 - \sum_{i=1}^n p_i b_i^2\right)^{1/2} \leq \left|ab - \sum_{i=1}^n p_i a_i b_i\right|,$$

assuming that

$$\sum_{i=1}^n p_i a_i^2 \leq a^2 \quad \text{and} \quad \sum_{i=1}^n p_i b_i^2 \leq b^2$$

and  $a_i, b_i, a, b \in R, 0 \leq p_i$  for  $i = 1, 2, \dots, n$ .

**Theorem 2.1.** *Assume that  $a_i, b_i, p_i, a, b$  are as above and  $0 \leq q_i \leq p_i$  for all  $i = 1, 2, \dots, n$ . Then we have the following inequality:*

$$(2.2) \quad \begin{aligned} 0 &\leq \sum_{i=1}^n q_i |a_i b_i| - \left| \sum_{i=1}^n q_i a_i b_i \right| \\ &\leq \left[ \sum_{i=1}^n q_i a_i^2 \sum_{i=1}^n q_i b_i^2 \right]^{1/2} - \left| \sum_{i=1}^n q_i a_i b_i \right| \\ &\leq \left| ab - \sum_{i=1}^n p_i a_i b_i \right| - \left( a^2 - \sum_{i=1}^n p_i a_i^2 \right)^{1/2} \left( b^2 - \sum_{i=1}^n p_i b_i^2 \right)^{1/2}. \end{aligned}$$

*Proof.* From  $a^2 - \sum_{i=1}^n p_i a_i^2 \geq 0$  and  $b^2 - \sum_{i=1}^n p_i b_i^2 \geq 0$ , it follows that

$$\begin{aligned} a^2 - \sum_{i=1}^n (p_i - q_i) a_i^2 &\geq a^2 - \sum_{i=1}^n p_i a_i^2 \geq 0, \\ b^2 - \sum_{i=1}^n (p_i - q_i) b_i^2 &\geq b^2 - \sum_{i=1}^n p_i b_i^2 \geq 0. \end{aligned}$$

Now, for  $t_i = p_i - q_i \geq 0$ , by (1.2), we have

$$\left(a^2 - \sum_{i=1}^n t_i a_i^2\right) \left(b^2 - \sum_{i=1}^n t_i b_i^2\right) \leq \left(ab - \sum_{i=1}^n t_i a_i b_i\right)^2,$$

i.e.,

$$\begin{aligned} & \left[ \left( a^2 - \sum_{i=1}^n p_i a_i^2 \right) + \sum_{i=1}^n q_i a_i^2 \right] \left[ \left( b^2 - \sum_{i=1}^n p_i b_i^2 \right) + \sum_{i=1}^n q_i b_i^2 \right] \\ & \leq \left[ \left( ab - \sum_{i=1}^n p_i a_i b_i \right) + \sum_{i=1}^n q_i a_i b_i \right]^2. \end{aligned}$$

Applying the well-known Cauchy-Buniakowski-Schwarz's inequality for real number, we have

$$\begin{aligned} (2.4) \quad & \left[ \left( \sum_{i=1}^n q_i a_i^2 \right)^{1/2} \left( \sum_{i=1}^n q_i b_i^2 \right)^{1/2} + \left( a^2 - \sum_{i=1}^n p_i a_i^2 \right)^{1/2} \left( b^2 - \sum_{i=1}^n p_i b_i^2 \right)^{1/2} \right]^2 \\ & \leq \left( \sum_{i=1}^n q_i a_i^2 + \left[ \left( a^2 - \sum_{i=1}^n p_i a_i^2 \right)^{1/2} \right]^2 \right) \left( \sum_{i=1}^n q_i b_i^2 + \left[ \left( b^2 - \sum_{i=1}^n p_i b_i^2 \right)^{1/2} \right]^2 \right) \end{aligned}$$

and, by the triangle inequality,

$$\begin{aligned} (2.5) \quad & \left| \left( ab - \sum_{i=1}^n p_i a_i b_i \right) + \sum_{i=1}^n q_i a_i b_i \right| \\ & \leq \left| ab - \sum_{i=1}^n p_i a_i b_i \right| + \left| \sum_{i=1}^n q_i a_i b_i \right|. \end{aligned}$$

Thus, from (2.3), (2.4) and (2.5), it follows that

$$\begin{aligned} & \left( \sum_{i=1}^n q_i a_i^2 \right)^{1/2} \left( \sum_{i=1}^n q_i b_i^2 \right)^{1/2} + \left( a^2 - \sum_{i=1}^n p_i a_i^2 \right)^{1/2} \left( b^2 - \sum_{i=1}^n p_i b_i^2 \right)^{1/2} \\ & \leq \left| ab - \sum_{i=1}^n p_i a_i b_i \right| + \left| \sum_{i=1}^n q_i a_i b_i \right|, \end{aligned}$$

which implies that

$$\begin{aligned} 0 & \leq \sum_{i=1}^n q_i |a_i b_i| - \left| \sum_{i=1}^n q_i a_i b_i \right| \\ & \leq \left( \sum_{i=1}^n q_i a_i^2 \right)^{1/2} \left( \sum_{i=1}^n q_i b_i^2 \right)^{1/2} - \left| \sum_{i=1}^n q_i a_i b_i \right| \\ & \leq \left| ab - \sum_{i=1}^n p_i a_i b_i \right| - \left( a^2 - \sum_{i=1}^n p_i a_i^2 \right)^{1/2} \left( b^2 - \sum_{i=1}^n p_i b_i^2 \right)^{1/2}. \end{aligned}$$

This completes the proof.

**Corollary 2.2.** *With the above assumptions for  $a_i, b_i, a, b \in \mathbb{R}$  and  $0 \leq p_i$  for  $i = 1, 2, \dots, n$ , we have the following inequality:*

$$\begin{aligned} 0 &\leq \sum_{i=1}^n p_i |a_i b_i| - \left| \sum_{i=1}^n p_i a_i b_i \right| \\ &\leq \left[ \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 \right]^{1/2} - \left| \sum_{i=1}^n p_i a_i b_i \right| \\ &\leq \left| ab - \sum_{i=1}^n p_i a_i b_i \right| - \left( a^2 - \sum_{i=1}^n p_i a_i^2 \right)^{1/2} \left( b^2 - \sum_{i=1}^n p_i b_i^2 \right)^{1/2}. \end{aligned}$$

Another result of Aczél's type is as follows:

**Theorem 2.3.** *Assume that  $a, b, a_i, b_i \in \mathbb{R}$  and  $0 \leq p_i$  for  $i = 1, 2, \dots, n$  are such that*

$$\sum_{i=1}^n p_i a_i^2 \leq a^2 \quad \text{and} \quad \sum_{i=1}^n p_i b_i^2 \leq b^2.$$

*Then we have the following inequality:*

$$\begin{aligned} &\left[ |a| - \left( \sum_{i=1}^n p_i a_i^2 \right)^{1/2} \right]^{1/2} \left[ |b| - \left( \sum_{i=1}^n p_i b_i^2 \right)^{1/2} \right]^{1/2} \\ &\leq |ab|^{1/2} - \left| \sum_{i=1}^n p_i a_i b_i \right|^{1/2}. \end{aligned}$$

*Proof.* We will start with the following elementary inequality:

$$(2.7) \quad \sqrt{(x-y)(z-u)} \leq \sqrt{xz} - \sqrt{yu},$$

where  $x \geq y \geq 0$  and  $z \geq u \geq 0$ . Indeed, the inequality (2.7) is equivalent with

$$(x-y)(z-u) \leq (\sqrt{xz} - \sqrt{yu})^2 = xz - 2\sqrt{xzyu} + yu,$$

i.e.,

$$xz + yu - yz - xu \leq xz - 2\sqrt{xzyu} + yu,$$

which is equivalent with

$$2\sqrt{xzyu} \leq yz + xu$$

for  $x, y, z, u \geq 0$ , which is obvious.

Now, putting

$$x = |a|, \quad y = \left( \sum_{i=1}^n p_i a_i^2 \right)^{1/2}, \quad z = |b|, \quad u = \left( \sum_{i=1}^n p_i b_i^2 \right)^{1/2},$$

then, by the inequality (2.7), we have

$$(2.8) \quad \begin{aligned} & \left[ |a| - \left( \sum_{i=1}^n p_i a_i^2 \right)^{1/2} \right]^{1/2} \left[ |b| - \left( \sum_{i=1}^n p_i b_i^2 \right)^{1/2} \right]^{1/2} \\ & \leq |ab|^{1/2} - \left( \sum_{i=1}^n p_i a_i^2 \right)^{1/4} \left( \sum_{i=1}^n p_i b_i^2 \right)^{1/4}. \end{aligned}$$

By Cauchy-Buniakowski-Schwarz's inequality, we have

$$\left| \sum_{i=1}^n p_i a_i b_i \right|^{1/2} \leq \left[ \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 \right]^{1/4}$$

and so, by (2.8), we have the desired inequality (2.6). This completes the proof.

**Corollary 2.4.** *Let  $a, b, a_i, b_i \in R$  for  $i = 1, 2, \dots, n$  be such that*

$$\sum_{i=1}^n a_i^2 \leq a^2 \quad \text{and} \quad \sum_{i=1}^n b_i^2 \leq b^2.$$

*Then we have the following inequality:*

$$\begin{aligned} & \left[ |a| - \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \right]^{1/2} \left[ |b| - \left( \sum_{i=1}^n b_i^2 \right)^{1/2} \right]^{1/2} \\ & \leq |ab|^{1/2} - \left| \sum_{i=1}^n a_i b_i \right|^{1/2}. \end{aligned}$$

**Remark.** The inequality (2.9) was proved in [1] as a particular case of an inequality holding in inner product spaces.

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