FURTHER INEQUALITIES FOR THE EXPECTATION AND VARIANCE OF A RANDOM VARIABLE DEFINED ON A FINITE INTERVAL

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ABSTRACT. Some new elementary inequalities for the expectation and the variance of a continuous random variable defined on a finite interval are given.

1. INTRODUCTION

Let X be a continuous random variable having the probability density function $f:[a,b]\to(0,\infty)$ and the cumulative distribution function $F:[a,b]\to[0,1]$.

In a recent paper [10], the authors pointed out a number of inequalities for the expectation, $E(X)$ and the variance, $\sigma^2(X)$ from which we cite only the following:

$$0\leq\sigma^2(X)\leq[b-E(X)][E(X)-a]\leq\frac{1}{4}(b-a)^2; \tag{1.1}$$

$$\begin{aligned} 0 &\leq [b-E(X)][E(X)-a]-\sigma^2(X) \\ &\leq \begin{cases} \frac{(b-a)^3}{6}\|f\|_\infty \\ [B(q+1,q+1)]^{\frac{1}{q}}(b-a)^{2+\frac{1}{q}}\|f\|_p, \\ \text{provided } f\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \end{cases} \end{aligned} \tag{1.2}$$

where $B(\cdot,\cdot)$ is Euler's Beta function, i.e., we recall it

$$B(\alpha,\beta):=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt,\ \ \alpha,\beta>-1.$$

Moreover, if $m\leq f\leq M$ a.e. on $[a,b]$, then

$$\frac{m(b-a)^3}{6}\leq[b-E(X)][E(X)-a]-\sigma^2(X)\leq\frac{M(b-a)^3}{6} \tag{1.3}$$

and

$$\left|[b-E(X)][E(X)-a]-\sigma^2(X)-\frac{(b-a)^3}{6}\right|\leq\frac{\sqrt{5}(b-a)^3(M-m)}{60}. \tag{1.4}$$

In this current paper, we point out some additional results.

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2. The Results

Lemma 1. *Let X be a continuous random variable having the cumulative distribution function $F:[a,b]\to[0,1]$. Then,*

$$\sigma^2(X) = (b-E(X))(E(X)-a) + \frac{1}{b-a}\int_a^b\int_a^b (t-\tau)(F(\tau)-F(t))\,d\tau dt. \tag{2.1}$$

Proof. Using integration by parts, we have

$$\begin{aligned}\sigma^2(X) &= \int_a^b (t-E(X))^2\,dF(t) \\ &= (t-E(X))^2 F(t)\Big|_a^b - 2\int_a^b (t-E(X))F(t)\,dt \\ &= (b-E(X))^2 - 2\int_a^b (t-E(X))F(t)\,dt.\end{aligned} \tag{2.2}$$

Further, using Korkine's identity,

$$\begin{aligned}\frac{1}{b-a}\int_a^b h(t)g(t)\,dt &= \frac{1}{b-a}\int_a^b h(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt \\ &\quad+\frac{1}{2(b-a)^2}\int_a^b\int_a^b (h(t)-h(\tau))(g(t)-g(\tau))\,d\tau dt,\end{aligned}$$

we have

$$\begin{aligned}&\int_a^b (t-E(X))F(t)\,dt \\ &= \frac{1}{b-a}\int_a^b (t-E(X))\,dt\int_a^b F(t)\,dt \\ &\quad+\frac{1}{2(b-a)}\int_a^b\int_a^b (t-\tau)(F(t)-F(\tau))\,d\tau dt.\end{aligned} \tag{2.3}$$

Since,

$$\begin{aligned}\int_a^b (t-E(X))\,dt &= \frac{(b-E(X))^2-(E(X)-a)^2}{2} \\ &= (b-a)\left(\frac{b+a}{2}-E(X)\right)\end{aligned}$$

and

$$\int_a^b F(t)\,dt = b-E(X),$$

then, by (2.2) and (2.3),

$$\begin{aligned}\sigma^2(X) &= (b-E(X))^2 - 2\left[\frac{b+a-2E(X)}{2}\cdot(b-E(X))\right.\\ &\quad\left.+\frac{1}{2(b-a)}\int_a^b\int_a^b (t-\tau)(F(t)-F(\tau))\,d\tau dt\right]\\ &= (b-E(X))^2-(b+a-2E(X))(b-E(X))\\ &\quad -\frac{1}{b-a}\int_a^b\int_a^b(t-\tau)(F(t)-F(\tau))\,d\tau dt\\ &= (b-E(X))(E(X)-a)-\frac{1}{b-a}\int_a^b\int_a^b(t-\tau)(F(t)-F(\tau))\,d\tau dt\end{aligned}$$

and the lemma is proved. ∎

Remark 1. *Since the mapping F is monotonic nondecreasing on $[a,b]$, then*

$$(t-\tau)(F(\tau)-F(t))\le 0 \quad \textit{for all} \quad t,\tau\in[a,b]; \tag{2.4}$$

which implies that

$$\sigma^2(X)\le[b-E(X)][E(X)-a], \tag{2.5}$$

an inequality that was proved in [10] *and* [11] *using two different methods.*

The inequality (2.5) can be improved as follows.

Theorem 1. *With the assumptions in Lemma 1,*

$$\begin{aligned}&(b-E(X))(E(X)-a)-\sigma^2(X)\\ &\ge 2\left|\int_a^b |t|\,F(t)\,dt-\frac{1}{b-a}(b-E(X))\int_a^b|t|\,dt\right|\ge 0.\end{aligned} \tag{2.6}$$

Proof. In [12], S.S. Dragomir proved the following refinement of Chebychev's inequality

$$T(h,g)\ge\max\{|T(h,|g|)|,|T(|h|,g)|,|T(|h|,|g|)|\}\ge 0, \tag{2.7}$$

provided (h,g) are synchronous on $[a,b]$, i.e.,

$$(h(t)-h(\tau))(g(t)-g(\tau))\ge 0 \quad\text{for all}\quad t,\tau\in[a,b]$$

and

$$T(h,g):=\frac{1}{b-a}\int_a^b h(t)g(t)\,dt-\frac{1}{b-a}\int_a^b h(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt.$$

If we define $h(t)=t$, $t\in[a,b]$, then from (2.1)

$$\begin{aligned}T(h,F) &= \frac{1}{b-a}\int_a^b\int_a^b(t-\tau)(F(t)-F(\tau))\,d\tau dt\\ &= \frac{1}{b-a}\left[(b-E(X))(E(X)-a)-\sigma^2(X)\right].\end{aligned}$$

Now, from (2.7),

$$\begin{aligned} T(|h|,F) &= \frac{1}{b-a}\int_a^b\int_a^b |t|\,F(t)\,dt - \frac{1}{(b-a)^2}\int_a^b |t|\,dt\int_a^b F(t)\,dt, \\ T(h,|F|) &= T(h,F), \\ T(|h|,|F|) &= T(|h|,F). \end{aligned}$$

Using the result (2.7), we get (2.6). ∎

Remark 2. *If $a \le b \le 0$ or $0 \le a \le b$, then the first inequality in (2.6) becomes an identity and is of no special interest.*
If $a < 0 < b$, however, then,

$$\begin{aligned} \int_a^b |t|\,F(t)\,dt &= -\int_a^0 tF(t)\,dt + \int_0^b tF(t)\,dt; \\ \frac{1}{b-a}\int_a^b |t|\,dt &= \frac{1}{b-a}\left[-\int_a^0 tdt + \int_0^b tdt\right] = \frac{1}{b-a}\left[\frac{a^2+b^2}{2}\right] \end{aligned}$$

and by (2.6), we get

$$\begin{aligned} &(b-E(X))(E(X)-a)-\sigma^2(X) \\ \ge\ & 2\left|\int_0^b tF(t)\,dt - \int_a^0 tF(t)\,dt - \frac{a^2+b^2}{2(b-a)}(b-E(X))\right| \ge 0. \end{aligned} \tag{2.8}$$

Assume that f, $f:[a,b]\to(0,\infty)$ is the p.d.f. of X, then the following theorem holds.

Theorem 2. *With the assumptions in Lemma 1,*

$$\begin{aligned} 0 &\le (b-E(X))(E(X)-a)-\sigma^2(X) \\ &\le \begin{cases} \frac{(b-a)^3}{6}\|f\|_\infty & \text{if} \quad f\in L_\infty[a,b]; \\ \frac{2q^2(b-a)^{2+\frac{1}{q}}\|f\|_p}{(2q+1)(3q+1)} & \text{if} \quad f\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^2}{3}; \end{cases} \end{aligned} \tag{2.9}$$

where $\|\cdot\|_p$ $(p\ge 1)$ on the usual Lebesgue norm.

Proof. Using (2.1),

$$\begin{aligned} 0 &\le (b-E(X))(E(X)-a)-\sigma^2(X) \\ &= \frac{1}{b-a}\int_a^b\int_a^b (t-\tau)\left(\int_\tau^t f(u)\,du\right)dtd\tau. \end{aligned} \tag{2.10}$$

By the modulus property, we have

$$\begin{aligned} 0 &\le (b-E(X))(E(X)-a)-\sigma^2(X) \\ &= \frac{1}{b-a}\left|\int_a^b\int_a^b (t-\tau)\left(\int_\tau^t f(u)\,du\right)dtd\tau\right| \\ &\le \frac{1}{b-a}\int_a^b\int_a^b |t-\tau|\left|\int_\tau^t f(u)\,du\right|dtd\tau =: M. \end{aligned} \tag{2.11}$$

If $f \in L_\infty[a,b]$, then we can write,

$$\left|\int_\tau^t f(u)\,du\right| \le |t-\tau|\,\|f\|_\infty$$

for all $t,\tau \in [a,b]$, and so

$$\begin{aligned} M &\le \frac{1}{b-a}\int_a^b\int_a^b |t-\tau|\,|t-\tau|\,\|f\|_\infty\,dtd\tau \\ &= \frac{\|f\|_\infty}{b-a}\int_a^b\int_a^b (t-\tau)^2\,dtd\tau = \frac{\|f\|_\infty (b-a)^3}{6}. \end{aligned}$$

For the second part, we apply Hölder's integral inequality to write:

$$\begin{aligned} \left|\int_\tau^t f(u)\,du\right| &\le \left|\int_\tau^t du\right|^{\frac{1}{q}}\left|\int_\tau^t f^p(u)\,du\right|^{\frac{1}{p}} \le |t-\tau|^{\frac{1}{q}}\left(\int_a^b f^p(u)\,du\right)^{\frac{1}{p}} \\ &= |t-\tau|^{\frac{1}{q}}\,\|f\|_p\,, \end{aligned}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

In addition,

$$\begin{aligned} M &\le \frac{1}{b-a}\int_a^b\int_a^b |t-\tau|\,|t-\tau|^{\frac{1}{q}}\,\|f\|_p\,dtd\tau \\ &= \frac{\|f\|_p}{b-a}\int_a^b\left[\int_a^t (t-\tau)^{1+\frac{1}{q}}\,d\tau + \int_t^b (\tau-t)^{1+\frac{1}{q}}\,d\tau\right]dt \\ &= \frac{\|f\|_p}{b-a}\int_a^b\left[\frac{(t-a)^{2+\frac{1}{q}}+(b-t)^{2+\frac{1}{q}}}{\left(2+\frac{1}{q}\right)}\right]dt = \frac{2\,\|f\|_p\,(b-a)^{2+\frac{1}{q}}}{\left(2+\frac{1}{q}\right)\left(3+\frac{1}{q}\right)} \end{aligned}$$

and the second inequality in (2.9) is proved.

Finally,

$$\begin{aligned} M &\le \frac{1}{b-a}\int_a^b\int_a^b |t-\tau|\left(\int_a^b f(u)\,du\right)dtd\tau \\ &= \frac{1}{b-a}\int_a^b\left[\frac{(t-a)^2+(b-t)^2}{2}\right]dt \\ &= \frac{1}{2(b-a)}\left[\frac{(b-a)^3}{3}+\frac{(b-a)^3}{3}\right] = \frac{(b-a)^2}{3}, \end{aligned}$$

and the theorem is completely proved. ∎

Using the Cauchy-Buniakowsky-Schwartz inequality, we have the following inequality.

Theorem 3. *If X and F are as in Lemma 1, then,*

$$\begin{aligned} 0 &\le (b-E(X))(E(X)-a)-\sigma^2(X) \\ &\le \frac{(b-a)^2}{\sqrt{3}}\left[(b-a)\,\|F\|_2^2-(b-E(X))^2\right]^{\frac{1}{2}} \\ &\le \frac{(b-a)^3}{2\sqrt{3}}. \end{aligned} \tag{2.12}$$

Proof. Using the Cauchy-Buniakowsky-Schwartz integral inequality for double integrals,

$$\left|\int_a^b \int_a^b (t-\tau)(F(\tau)-F(t))\, dt d\tau\right| \leq \left(\int_a^b \int_a^b (t-\tau)^2\, dt d\tau\right)^{\frac{1}{2}} \left(\int_a^b \int_a^b (F(t)-F(\tau))^2\, dt d\tau\right)^{\frac{1}{2}}. \tag{2.13}$$

However,

$$\begin{aligned} \int_a^b \int_a^b (t-\tau)^2\, dt d\tau &= \frac{(b-a)^4}{6}, \\ \int_a^b \int_a^b (F(\tau)-F(t))^2\, dt d\tau &= 2\left[(b-a)\int_a^b F^2(t)\, dt - \left(\int_a^b F(t)\, dt\right)^2\right] \\ &= 2\left[(b-a)\|F\|_2^2 - (b-E(X))^2\right] \end{aligned}$$

and, by (2.13),

$$\left|\int_a^b \int_a^b (t-\tau)(F(\tau)-F(t))\, d\tau dt\right| \leq \frac{(b-a)^2}{\sqrt{3}}\left[(b-a)\|F\|_2^2 - (b-E(X))^2\right]^{\frac{1}{2}}$$

and the first inequality in (2.12) is proved.

To prove the last part of (2.12), we use the following Grüss type inequality:

$$\frac{1}{b-a}\int_a^b g^2(t)\, dt - \left(\frac{1}{b-a}\int_a^b g(t)\, dt\right)^2 \leq \frac{1}{4}(\phi-\gamma)^2, \tag{2.14}$$

provided that $g \in L_2(a,b)$ and $\gamma \leq g(t) \leq \phi$ a.e. for $t \in (a,b)$.

From (2.14),

$$(b-a)\int_a^b F^2(t)\, dt - \left(\int_a^b F(t)\, dt\right)^2 \leq \frac{1}{4}(b-a)^2$$

since

$$\sup_{t\in[a,b]} F(t) = 1 \quad \text{and} \quad \inf_{t\in[a,b]} F(t) = 0.$$

∎

If it is assumed that the mapping f is convex on $[a,b]$, then the following result can be obtained.

Theorem 4. *Assume that the p.d.f.,* $f : [a,b] \to (0,\infty)$ *is convex. Then we have the inequality*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b \int_a^b (t-\tau)^2 f\left(\frac{t+\tau}{2}\right) d\tau dt \\ \leq\; & [b-E(X)][E(X)-a] - \sigma^2(X) \\ \leq\; & \frac{(b-a)^2}{3} + \sigma^2(X) - (b-E(X))(E(X)-a). \end{aligned} \tag{2.15}$$

Proof. Using the Hermite-Hadamard inequality,

$$f\left(\frac{t+\tau}{2}\right) \leq \frac{\int_t^\tau f(u)\,du}{\tau - t} \leq \frac{f(t)+f(\tau)}{2} \tag{2.16}$$

for all $t, \tau \in [a,b]$, $t \neq \tau$, we have

$$(t-\tau)^2 f\left(\frac{t+\tau}{2}\right) \leq (t-\tau)(F(t)-F(\tau)) \leq \frac{f(t)+f(\tau)}{2}(t-\tau)^2 \tag{2.17}$$

for all $t, \tau \in [a,b]$.

Integrating (2.17) on $[a,b]^2$ and using the representation (2.1), gives:-

$$\begin{aligned} &\frac{1}{b-a}\int_a^b\int_a^b (t-\tau)^2 f\left(\frac{t+\tau}{2}\right) dt d\tau \\ \leq\ & \frac{1}{b-a}\int_a^b\int_a^b (t-\tau)(F(t)-F(\tau))\,dt d\tau \\ =\ & [b-E(X)][E(X)-a]-\sigma^2(X) \\ \leq\ & \frac{1}{b-a}\int_a^b\int_a^b \frac{f(t)+f(\tau)}{2}(t-\tau)^2\,dt d\tau. \end{aligned} \tag{2.18}$$

Now

$$\begin{aligned} &\int_a^b\int_a^b (t-\tau)^2\left[\frac{f(t)+f(\tau)}{2}\right] dt d\tau \\ =\ & \int_a^b\int_a^b (t-\tau)^2 f(t)\,d\tau dt = \int_a^b\left[\int_a^b (t-\tau)^2\,d\tau\right] f(t)\,dt \\ =\ & \int_a^b \left[\frac{(b-t)^3+(t-a)^3}{3}\right] f(t)\,dt \\ =\ & \frac{(b-a)}{3}\int_a^b\left[(b-t)^2-(b-t)(t-a)+(t-a)^2\right] f(t)\,dt \\ =\ & \frac{b-a}{3}\int_a^b\left[(b-a)^2-3(b-t)(t-a)\right] f(t)\,dt \\ =\ & \frac{(b-a)^3}{3}-(b-a)\int_a^b (b-t)(t-a)f(t)\,dt \\ =\ & \frac{(b-a)^3}{3}+(b-a)\left[\sigma^2(X)-(b-E(X))(E(X)-a)\right] \end{aligned} \tag{2.19}$$

on using an identity (see [10]).

Hence, from (2.19) and the above working,

$$\begin{aligned} &\frac{1}{b-a}\int_a^b\int_a^b (t-\tau)^2\left[\frac{f(t)+f(\tau)}{2}\right] dt d\tau \\ =\ & \frac{(b-a)^2}{3}+\left[\sigma^2(X)-(b-E(X))(E(X)-a)\right], \end{aligned}$$

and the second part of (2.15) is proved. ∎

Remark 3. *The second inequality in (2.15) is equivalent to:-*

$$[b-E(X)][E(X)-a]\leq\sigma^{2}(X)+\frac{1}{6}(b-a)^{2}. \tag{2.20}$$

Remark 4. *For $b-a<\frac{1}{\sqrt{3}}$, then the result of Theorem 5 is better than that of Theorem 4. For $b-a>\frac{1}{\sqrt{3}}$, the opposite applies. It must be remembered that Theorem 4 relies on f being convex whereas Theorem 3 does not.*

The following representation for the absolutely continuous p.d.f., $f:[a,b]\to\mathbb{R}$ holds.

Lemma 2. *Let X be a random variable having the p.d.f., $f:[a,b]\to\mathbb{R}$ absolutely continuous on $[a,b]$. Then we have*

$$\begin{aligned}\sigma^{2}(X) &= (b-E(X))(E(X)-a)-\frac{(b-a)^{2}}{6}\\ &\quad+\frac{1}{2(b-a)}\int_{a}^{b}\int_{a}^{b}(t-\tau)\left(\int_{\tau}^{t}\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right)dtd\tau.\end{aligned} \tag{2.21}$$

Proof. We use the following identity which holds for the absolutely continuous mapping $g:[a,b]\to\mathbb{R}$

$$\int_{a}^{b}g(u)\,du=\frac{g(a)+g(b)}{2}(b-a)-\int_{a}^{b}\left(u-\frac{a+b}{2}\right)g'(u)\,du, \tag{2.22}$$

can be easily proven by using the integration by parts formula.

We know that

$$\begin{aligned}&(E(X)-a)(b-E(X))-\sigma^{2}(X) \\ &= \frac{1}{b-a}\int_{a}^{b}\int_{a}^{b}(t-\tau)\int_{\tau}^{t}f(u)\,du\,dtd\tau\\ &= \frac{1}{b-a}\int_{a}^{b}\int_{a}^{b}(t-\tau)\left[\frac{f(t)+f(\tau)}{2}(t-\tau)-\int_{\tau}^{t}\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right]dtd\tau\\ &= \frac{1}{b-a}\int_{a}^{b}\int_{a}^{b}(t-\tau)^{2}\left(\frac{f(t)+f(\tau)}{2}\right)dtd\tau\\ &\quad-\frac{1}{b-a}\int_{a}^{b}\int_{a}^{b}(t-\tau)\left(\int_{\tau}^{t}\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right)dtd\tau.\end{aligned} \tag{2.23}$$

However, observe that (see the proof of Theorem 4)

$$\begin{aligned}L &:= \frac{1}{b-a}\int_{a}^{b}\int_{a}^{b}(t-\tau)^{2}\left(\frac{f(t)+f(\tau)}{2}\right)dtd\tau\\ &\quad\sigma^{2}(X)+\frac{1}{3}\left[(E(X)-b)^{2}-(E(X)-a)(b-E(X))+(E(X)-a)^{2}\right].\end{aligned}$$

Using (2.23), we have

$$\begin{aligned}&(E(X)-a)(b-E(X))-\sigma^{2}(X)\\ &= \sigma^{2}(X)+\frac{1}{3}\left[(E(X)-b)^{2}-(E(X)-a)(b-E(X))+(E(X)-a)^{2}\right]\\ &\quad-\frac{1}{b-a}\int_{a}^{b}\int_{a}^{b}(t-\tau)\left(\int_{\tau}^{t}\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right)dtd\tau,\end{aligned}$$

which is clearly equivalent to (2.21). ∎

Using Lemma 2, we are able to obtain the following bounds.

Theorem 5. *Assume that f is as in Lemma 2. Then we have the inequality*

$$\left|[b-E(X)][E(X)-a]-\sigma^2(X)-\frac{(b-a)^2}{6}\right| \tag{2.24}$$
$$\leq\begin{cases}\frac{\|f'\|_\infty}{80}(b-a)^4 & \textit{if}\quad f'\in L_\infty[a,b];\\ \frac{q^2\|f'\|_p}{2(3q+1)(4q+1)(q+1)^{\frac{1}{q}}}(b-a)^{3+\frac{1}{q}} & \textit{if}\quad f'\in L_p[a,b],\\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \frac{\|f'\|_1}{24}(b-a)^3. \end{cases}$$

Proof. Using the equality (2.21), we may write

$$\left|\sigma^2(X)-(b-E(X))(E(X)-a)+\frac{(b-a)^2}{6}\right|$$
$$\leq \frac{1}{2(b-a)}\int_a^b\int_a^b|t-\tau|\left|\int_\tau^t\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right|dtd\tau:=N. \tag{2.25}$$

Now, it may be easily shown that,

$$\begin{aligned}\left|\int_\tau^t\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right| &\leq \|f'\|_\infty\left|\int_\tau^t\left|u-\frac{t+\tau}{2}\right|du\right|\\ &= \|f'\|_\infty\frac{(t-\tau)^2}{4}\end{aligned}$$

for all $t,\tau\in[a,b]$.

Also, by Hölder's integral inequality, we may write

$$\begin{aligned}\left|\int_\tau^t\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right| &\leq \left|\int_\tau^t|f'(u)|^p\,du\right|^{\frac{1}{p}}\left|\int_\tau^t\left|u-\frac{t+\tau}{2}\right|^q du\right|^{\frac{1}{q}}\\ &\leq \|f'\|_p\left[\frac{|t-\tau|^{q+1}}{2q(q+1)}\right]^{\frac{1}{q}}=\|f'\|_p\frac{|t-\tau|^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\end{aligned}$$

for all $t,\tau\in[a,b]$, and further,

$$\begin{aligned}\left|\int_\tau^t\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right| &\leq \sup\left|u-\frac{t+\tau}{2}\right|\int_\tau^t|f'(u)|\,du\\ &\leq \frac{|t-\tau|}{2}\|f'\|_1.\end{aligned}$$

Consequently,

$$\left|\int_\tau^t\left(u-\frac{t+\tau}{2}\right)f'(u)\,du\right|\leq\begin{cases}\|f'\|_\infty\frac{(t-\tau)^2}{4} & \text{if}\quad f'\in L_\infty[a,b];\\ \|f'\|_p\frac{|t-\tau|^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}} & \text{if}\quad f'\in L_p[a,b],\\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \|f'\|_1\frac{|t-\tau|}{2}. & \text{if}\quad f'\in L_1[a,b].\end{cases} \tag{2.26}$$

Using (2.26), we may write, for f' belonging to the obvious Lebesgue space $L_p[a,b]$, $p \geq 1$,

$$N \leq \begin{cases} \frac{\|f'\|_\infty}{8(b-a)} \int_a^b \int_a^b |t-\tau|^3 \, dt d\tau, \\ \frac{\|f'\|_p}{4(q+1)^{\frac{1}{q}}(b-a)} \int_a^b \int_a^b |t-\tau|^{2+\frac{1}{q}} \, dt d\tau, \\ \frac{\|f'\|_1}{4(b-a)} \int_a^b \int_a^b (t-\tau)^2 \, dt d\tau. \end{cases} \tag{2.27}$$

Now, since some straight forward algebra shows that

$$\begin{aligned} \int_a^b \int_a^b |t-\tau|^3 \, dt d\tau &= \int_a^b \left[\int_a^t (t-\tau)^3 \, d\tau + \int_t^b (\tau - t)^3 \, d\tau \right] dt \\ &= \int_a^b \left[\frac{(t-a)^4 + (b-t)^4}{4} \right] dt = \frac{(b-a)^5}{10}, \end{aligned}$$

$$\begin{aligned} \int_a^b \int_a^b |t-\tau|^{2+\frac{1}{q}} \, dt d\tau &= \int_a^b \left[\int_a^t (t-\tau)^{2+\frac{1}{q}} \, d\tau + \int_t^b (\tau - t)^{2+\frac{1}{q}} \, d\tau \right] dt \\ &= \int_a^b \left[\frac{(t-a)^{3+\frac{1}{q}} + (b-t)^{3+\frac{1}{q}}}{3+\frac{1}{q}} \right] dt \\ &= \frac{2q^2 (b-a)^{4+\frac{1}{q}}}{(3q+1)(4q+1)} \end{aligned}$$

and

$$\begin{aligned} \int_a^b \int_a^b (t-\tau)^2 \, dt d\tau &= \int_a^b \left[\int_a^t (t-\tau)^2 \, d\tau + \int_t^b (\tau - t)^2 \, d\tau \right] dt \\ &= \int_a^b \left[\frac{(t-a)^3 + (b-t)^3}{3} \right] dt \\ &= \frac{(b-a)^4}{6}, \end{aligned}$$

we obtain the desired inequality (2.24) from using (2.27) and (2.25). ∎

The following representation for the mappings whose derivatives are absolutely continuous on $[a,b]$ also holds.

Lemma 3. *Let X be a random variable having the p.d.f. $f : [a,b] \to \mathbb{R}$ and with the property that $f' : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$. Then we have*

$$\begin{aligned} \sigma^2(X) &= (b - E(X))(E(X) - a) - \frac{(b-a)^2}{6} \\ &+ \frac{1}{4(b-a)} \int_a^b \int_a^b (t-\tau) \int_\tau^t (t-u)(u-\tau) f''(u) \, du \, dt \, d\tau. \end{aligned} \tag{2.28}$$

Proof. We use the following identity which holds for the mappings g whose derivatives are absolutely continuous:

$$\int_a^b g(u)\,du = \frac{g(a)+g(b)}{2}(b-a) - \frac{1}{2}\int_a^b (b-u)(u-a)\,g''(u)\,du \tag{2.29}$$

and can easily be proven by using the integration by parts formula twice.

We know that

$$(b-E(X))(E(X)-a)-\sigma^2(X) = \frac{1}{b-a}\int_a^b\int_a^b (t-\tau)\int_t^\tau f(u)\,du\,dt\,d\tau$$

and then, using the representation (2.29) writen for f instead of g, and proceeding as in the proof of Lemma 2, we end up with the identity (2.28). ∎

Using the representation of Lemma 3, we are able to obtain the following bounds.

Theorem 6. *Assume that f is as in Lemma 3. Then we have the inequality*

$$\left|[b-E(X)][E(X)-a]-\sigma^2(X)-\frac{(b-a)^2}{6}\right| \tag{2.30}$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{360}(b-a)^5 & \text{if } f''\in L_\infty[a,b] \\[2mm] \frac{\|f''\|_q}{2(4p+1)(5p+1)}[B(p+1,p+1)]^{\frac{1}{p}}(b-a)^{4+\frac{1}{p}} & \text{if } f''\in L_q[a,b], \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\[2mm] \frac{\|f''\|_1}{160}(b-a)^4, & \end{cases}$$

where the $p-$norms are taken on the interval $[a,b]$.

Proof. Using the equality (2.28), we may write

$$\left|\sigma^2(X)-[b-E(X)][E(X)-a]-\frac{(b-a)^2}{6}\right|$$
$$\leq \frac{1}{4(b-a)}\int_a^b\int_a^b |t-\tau|\left|\int_\tau^t (t-u)(u-\tau)f''(u)\,du\right|dt\,d\tau := K.$$

First of all, let us observe that

$$\begin{aligned}\left|\int_\tau^t (t-u)(u-\tau)f''(u)\,du\right| &\leq \|f''\|_\infty\left|\int_\tau^t (t-u)(u-\tau)\,du\right| \\ &\leq \frac{\|f''\|_\infty}{6}|t-\tau|^3,\end{aligned}$$

for all $t,\tau\in[a,b]$.

Further, by Holder's integral inequality, we obtain

$$\begin{aligned}\left|\int_\tau^t (t-u)(u-\tau)f''(u)\,du\right| &\leq \|f''\|_q\left|\int_\tau^t |t-u|^p|u-\tau|^p\,du\right|^{\frac{1}{p}} \\ &= \|f''\|_q|t-\tau|^{2+\frac{1}{p}}[B(p+1,p+1)]^{\frac{1}{p}}\end{aligned}$$

for all $t,\tau\in[a,b]$, where B is the Beta function of Euler and $\frac{1}{p}+\frac{1}{q}=1$; $p>1$.

Also, we have

$$\left|\int_{\tau}^{t}(t-u)(u-\tau)f''(u)\,du\right| \leq \|f''\|_1 \max_{u\in[\tau,t]}|(t-u)(u-\tau)| = \frac{|t-\tau|^2}{4}\|f''\|_1$$

for all $t,\tau\in[a,b]$.

Consequently, we may state the inequality

$$\left|\int_{\tau}^{t}(t-u)(u-\tau)f''(u)\,du\right| \leq \begin{cases} \frac{\|f''\|_\infty}{6}|t-\tau|^3 & \text{if } f''\in L_\infty[a,b]; \\ \|f''\|_q[B(p+1,p+1)]^{\frac{1}{p}}|t-\tau|^{2+\frac{1}{p}} & \text{if } f''\in L_q[a,b], \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \frac{|t-\tau|^2}{4}\|f''\|_1, \end{cases} \tag{2.31}$$

for all $t,\tau\in[a,b]$.

Using (2.31) and the definition of K above, we may write

$$K \leq \begin{cases} \frac{\|f''\|_\infty}{24(b-a)}\int_a^b\int_a^b(t-\tau)^4\,dt\,d\tau & \text{if } f''\in L_\infty[a,b]; \\ \frac{\|f''\|_q}{4(b-a)}[B(p+1,p+1)]^{\frac{1}{p}}\int_a^b\int_a^b|t-\tau|^{3+\frac{1}{p}}\,dt\,d\tau & \text{if } f''\in L_q[a,b], \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \frac{\|f''\|_1}{16(b-a)}\int_a^b\int_a^b|t-\tau|^3\,dt\,d\tau. \end{cases} \tag{2.32}$$

Now, since some straight forward algebra shows that

$$\int_a^b\int_a^b(t-\tau)^4\,dt\,d\tau = \frac{(b-a)^6}{15}.$$

$$\begin{aligned}\int_a^b\int_a^b|t-\tau|^{3+\frac{1}{p}}\,dt\,d\tau &= \int_a^b\left[\int_a^t(t-\tau)^{3+\frac{1}{p}}\,d\tau+\int_t^b(\tau-t)^{3+\frac{1}{p}}\,d\tau\right]dt \\ &= \int_a^b\left[\frac{(t-a)^{4+\frac{1}{p}}+(b-t)^{4+\frac{1}{p}}}{4+\frac{1}{p}}\right]dt \\ &= 2\frac{(b-a)^{5+\frac{1}{p}}}{\left(4+\frac{1}{p}\right)\left(5+\frac{1}{p}\right)} = \frac{2p^2(b-a)^{5+\frac{1}{p}}}{(4p+1)(5p+1)}\end{aligned}$$

and

$$\begin{aligned}\int_a^b\int_a^b|t-\tau|^3\,dt\,d\tau &= \int_a^b\left[\int_a^t(t-\tau)^3\,d\tau+\int_t^b(\tau-t)^3\,d\tau\right]dt \\ &= \int_a^b\left[\frac{(t-a)^4+(b-t)^4}{4}\right]dt = \frac{(b-a)^5}{10},\end{aligned}$$

then by (2.32), we deduce the desired inequality (2.30). ∎

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