ON NUMERICAL EVALUATION OF THE WINDING NUMBER OF A PLANE VECTOR FIELD

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Abstract. In this paper we prove a new inequality of Grüss type for the Riemann-Stieltjes line integral, which gives the possibility of numerically evaluating line integrals of continuous Lipschitzian functions.

1. Introduction

The Grüss inequality may be stated as [2]

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx\right| \leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \tag{1.1}$$

where f and g are two integrable functions on $[a,b]$ satisfying the condition $\phi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a,b]$. This result provides us with a tool for establishing a large number of new results in different domains of mathematics. In particular, in applications of numerical quadrature methods theory, c.f., for example [3].

The constant $\frac{1}{4}$ in (1.1) is the *best possible one* and is achieved for

$$f(x) = g(x) = sgn\left(x - \frac{a+b}{2}\right).$$

In this paper we point out a Grüss type inequality for the Riemann-Stieltjes integral and apply it to the numerical evaluation of the Poincaré integral

$$I \equiv \frac{1}{2\pi}\int_{AB} \frac{\varphi(x,y)d\psi(x,y) - \psi(x,y)d\varphi(x,y)}{\varphi^2(x,y)+\psi^2(x,y)}, \tag{1.2}$$

where $\varphi(x,y)$ and $\psi(x,y)$ are Lipschitz-continuous functions, AB is a smooth Jordan curve on $\mathbb{R}^2$, and the integral (1.2) is taken as the Riemann-Stieltjes integral.

Consider on the plane a Lipschitz-continuous vector field

$$\mathbf{F}(x,y) = \{\varphi(x,y)\ ,\ \psi(x,y)\}.$$

Formula (1.2) allows us to evaluate the rotation of this field with respect to the curve AB. Let us suppose that $\mathbf{F}$ has an isolated singularity point at the origin.

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This implies that $\varphi(0,0) = \psi(0,0) = 0$. Here, we can take as the curve AB, a circle of sufficiently small radius. Without loss of generality it can be the circle of radius 1.

In this case the particularity of the problem is due to the fact that the Poincaré integral is equal to an integer. Therefore the accuracy in the evaluation of integral (1.2) is given by a number less than $\frac{1}{2}$. The value of integral (1.2) over the unit circle is called *winding number* or *index* of the singular point and the value of this number represents the most important property of the vector field $\mathbf{F}$. For example, the knowledge of the index is crucial in the investigation of a non-linear dynamic system of the form

$$\begin{cases} \dot{x} = \varphi(x,y) \\ \dot{y} = \psi(x,y) \end{cases}.$$

In general, the exact computation in the integral (1.2) is very complicated and from this point of view, the Poincaré formula is only of theoretical interest [1]. The numerical integration can also be excessively complicated due to the following reasons. In the case of even very simple fields, for example, for the linear fields with the singularity of nodal type, the function $\dfrac{\varphi(x,y)}{\varphi^2(x,y)+\psi^2(x,y)}$ can have a very large value for the first derivative (of order 10^2 - 10^3), not to mention a derivative of order 4. For these reasons, even application of the Simpson rule is impractical.

2. Grüss inequalities for the Riemann-Stieltjes Line Integral

Let $f, u : AB \to \mathbb{R}$ be Lipschitzian on the curve AB with the Lipschitz constants L_f and L_u respectively. That is,

$$|f(A_1) - f(A_2)| \le L_f S(A_1A_2), \tag{2.1}$$

$$|u(A_1) - u(A_2)| \le L_u S(A_1A_2), \tag{2.2}$$

for all $A_1(x_1,y_1)$ and $A_1(x_2,y_2) \in AB$, where $S(A_1A_2)$ is the length of arc A_1A_2.

It is easy to see that *if f is an L_f-Lipschitzian mapping and u is Riemann integrable on AB, then*

$$\left| \int_{AB} u(x,y)df(x,y) \right| \le L_f \int_{AB} |u(x,y)|ds. \tag{2.3}$$

Indeed, if

$$\Delta_n = \left\{ A = \left(x_0^{(n)}, y_0^{(n)}\right), ..., A_i^{(n)} = \left(x_i^{(n)}, y_i^{(n)}\right), ..., \left(x_n^{(n)}, y_n^{(n)}\right) = B \right\}$$

is a sequence of partitions of AB, corresponding to the positive direction introduced on AB with $\lambda(\Delta_n) = \max_{i=\overline{1,n}} \Delta S_i^{(n)} \to 0$ (for $n \to \infty$) , $M_i^{(n)}\left(\xi_i^{(n)}, \eta_i^{(n)}\right) \in A_i^{(n)} A_{i-1}^{(n)}$

and $\Delta S_i^{(n)}=$ length of arc $A_i^{(n)}A_{i-1}^{(n)}$ then

$$\begin{aligned}\left|\int_{AB} u(x,y)df(x,y)\right| &= \left|\lim_{\lambda(\Delta_n)\to 0}\sum_{i=1}^{n} u\left(M_i^{(n)}\right)\left[f(A_i^{(n)})-f(A_{i-1}^{(n)})\right]\right| \\ &\leq \lim_{\lambda(\Delta_n)\to 0}\sum_{i=1}^{n}\left|u(M_i^{(n)})\right|\left|\frac{f(A_i^{(n)})-f(A_{i-1}^{(n)})}{\Delta S_i^{(n)}}\right|\Delta S_i^{(n)} \\ &\leq L\lim_{f\lambda(\Delta_n)\to 0}\sum_{i=1}^{n}\left|u(M_i^{(n)})\right|\Delta S_i^{(n)} = L_f\int_{AB}|u(x,y)|ds.\end{aligned}$$

The following result of Grüss type holds:

Lemma 1. *Let $f,u:AB\to\mathbb{R}$ be such that f is L_f-Lipschitzian on AB and u is Riemann integrable on the arc AB whose length is equal to S, and so that there exist the real numbers m,M such that*

$$m\leq u(x,y)\leq M,\ \textit{for all}\ (x,y)\in AB. \tag{2.4}$$

Then the following inequality holds

$$\left|\int_{AB} u(x,y)df(x,y)-\frac{f(B)-f(A)}{S}\int_{AB} u(x,y)ds\right|\leq\frac{1}{2}L_f(M-m)S \tag{2.5}$$

and the constant $\frac{1}{2}$ is sharp, in the sense that it cannot be replaced by a smaller one.

Proof. Let us denote $\frac{1}{S}\int_{AB} u(x,y)ds\equiv\overline{u}$ and consider the difference

$$\int_{AB} u(x,y)df(x,y)-\left[f(B)-f(A)\right]\overline{u}.$$

Since $\int_{AB}\overline{u}df(x,y)=\left[f(B)-f(A)\right]\overline{u}$, then by (2.3)

$$\begin{aligned}\left|\int_{AB} u(x,y)df(x,y)-\left[f(B)-f(A)\right]\overline{u}\right| &= \left|\int_{AB}\left[u(x,y)-\overline{u}\right]df(x,y)\right| \\ &\leq L\left\|u(x,y)-\overline{u}\right\|_1,\end{aligned}$$

where $\left\|\cdot\right\|_l=\left[\int_{AB}|\cdot|^l\,dt\right]^{\frac{1}{l}}$, $l\in[1,\infty)$.

In addition, as for finite AB we have $\left\|\cdot\right\|_1\leq\sqrt{S}\left\|\cdot\right\|_2$, we can write

$$\begin{aligned}\left|\int_{AB} u(x,y)df(x,y)-\left[f(B)-f(A)\right]\overline{u}\right| &\leq L\left\|u(x,y)-\overline{u}\right\|_1 \\ &\leq L\sqrt{S}\left\|u(x,y)-\overline{u}\right\|_2\end{aligned} \tag{2.6}$$

and we have to evaluate the norm $\|u(x,y)-\overline{u}\|_2$. We have

$$\frac{1}{S}\|u(x,y)-\overline{u}\|_2^2=\frac{1}{S}\int\limits_{AB}[u(x,y)-\overline{u}]^2\,ds=\frac{1}{S}\|u\|_2^2-\overline{u}^2. \tag{2.7}$$

By the inequality (2.4), we obtain

$$\begin{aligned} 0 &\leq (M-u(x,y))(u(x,y)-m) \\ &= Mu(x,y)-Mm-u^2(x,y)+mu(x,y). \end{aligned}$$

After integration and division by S we obtain

$$M\overline{u}-Mm+m\overline{u}\geq\frac{1}{S}\|u\|^2,$$

which implies that

$$\frac{1}{S}\|u\|^2-\overline{u}^2\leq M\overline{u}-Mm+m\overline{u}-\overline{u}^2=(M-\overline{u})(\overline{u}-m).$$

Using the elementary inequality

$$(M-\overline{u})(\overline{u}-m)\leq\frac{1}{4}[(M-\overline{u})+(\overline{u}-m)]^2=\frac{1}{4}(M-m)^2$$

which holds for $\overline{u},m,M\in R$, we arrive at

$$\frac{1}{S}\|u\|^2-\overline{u}^2\leq(M-\overline{u})(\overline{u}-m)\leq\frac{1}{4}(M-m)^2. \tag{2.8}$$

Combining (2.6), (2.7) and (2.8), we obtain the desired inequality (2.5).
The proof of sharpness of the constant $\frac{1}{2}$ is similar to the one in [3] after parametrization of AB and we omit the details. ∎

The term $\dfrac{f(B)-f(A)}{S}\displaystyle\int\limits_{AB}u(x,y)ds$ in (2.5) can be replaced by a simpler one, as shown in the following Lemma 3. To do this, we use the next simpler result.

Lemma 2. *Let $u:AB\to\mathbb{R}$ be L_u-Lipschitzian on AB. Then the following inequality holds*

$$\left|\int\limits_{AB}u(x,y)ds-Su(\xi,\eta)\right|\leq\frac{L_u}{4}S^2, \tag{2.9}$$

where the point (ξ,η) bisects the arc AB.

Proof. Let us introduce on AB the parametrization with the arc length s as a parameter $(0\leq s\leq S)$. Let

$$p(s)=s-\theta^+(s-\sigma)S$$

be a function defined on AB. Consider the integral $I=\displaystyle\int\limits_{AB}p(s)du(s)$.

Integrating by parts, using

$$p'(s)=1-S\delta(s-\sigma),$$

we obtain

$$\int\limits_{AB}p(s)du(s)=-\int\limits_{AB}u(s)p'(s)ds=-\int\limits_{AB}u(s)ds+Su(\sigma),$$

which implies by (2.3) that

$$\begin{aligned}&\left|\int\limits_{AB} u(s)ds - Su(\sigma)\right| \\ \leq\ & L_u \int\limits_{AB} |p(s)|\, ds = L_u \left[\left|\left(\frac{\sigma^2}{2} + \frac{(\sigma-S)^2}{2}\right)\right|_{\sigma=\frac{1}{2}S}\right] \\ =\ & \frac{L_u}{4}S^2,\end{aligned}$$

and the lemma is proved. ∎

Lemma 3. *Let $f, u : AB \to \mathbb{R}$ be such that f is L_f-Lipschitzian on AB and u is L_u-Lipschitzian on AB. Then the following inequality holds*

$$\left|\int\limits_{AB} u(x,y)df(x,y) - [f(B)-f(A)]\, u(\xi,\eta)\right| \leq \frac{3L_f L_u}{4}S^2, \tag{2.10}$$

where (ξ, η) is the point dividing the arc AB in two.

Proof. Let us denote by m, M the real numbers such that

$$m \leq u(x,y) \leq M, \text{ for all } (x,y) \in AB, \tag{2.11}$$

with the assumption that $M = u(A_M)$ and $m = u(A_m)$.

It is clear that

$$M - m = u(A_M) - u(A_m) \leq L_u S(A_M A_m) \leq L_u S. \tag{2.12}$$

Using the triangle inequality, Lemmas 1, 2 and (2.12), we obtain

$$\begin{aligned}&\left|\int\limits_{AB} u(x,y)df(x,y) - [f(B)-f(A)]\, u(\xi,\eta)\right| \\ =\ & \Bigg|\int\limits_{AB} u(x,y)df(x,y) \\ & - [f(B)-f(A)]\left[\frac{1}{S}\int\limits_{AB} u(x,y)ds + \left(u(\xi,\eta) - \frac{1}{S}\int\limits_{AB} u(x,y)ds\right)\right]\Bigg| \\ \leq\ & \left|\int\limits_{AB} u(x,y)df(x,y) - \frac{f(B)-f(A)}{S}\int\limits_{AB} u(x,y)ds\right| \\ & + \frac{|f(B)-f(A)|}{S}\left|Su(\xi) - \int\limits_{AB} u(x,y)ds\right| \\ \leq\ & \frac{L_f}{2}(M-m)S + \frac{L_f L_u}{4}S^2 \leq \frac{3L_f L_u}{4}S^2.\end{aligned}$$

Hence the lemma is proved. ∎

3. Application of the Grüss Inequalities to Adaptive Quadrature Methods.

Consider the Riemann-Stieltjes line integral

$$I(u,f)=\int\limits_{AB} u(x,y)df(x,y), \tag{3.1}$$

where $u(x,y)$ and $f(x,y)$ are Lipschitz-continuous on the smooth Jordan curve AB. Let the set of points $\{A=A_0,A_1,...,A_{n-1},A_n=B\}$ belonging to AB represent a subdivision of arc AB. Also, let (x_i,y_i) be coordinates of the point A_i. We consider the following approximation of $I(u,f)$ with the functional

$$I_n(u,f)=\sum_{i=1}^{n}\left[f(x_i,y_i)-f(x_{i-1},y_{i-1})\right]u(\xi_i,\eta_i).$$

Therefore,

$$\int\limits_{AB} u(x,y)df(x,y)\cong\sum_{i=1}^{n}\left[f(x_i,y_i)-f(x_{i-1},y_{i-1})\right]u(\xi_i,\eta_i),$$

where (ξ_i,η_i) is the midpoint of the arc $(A_{i-1}A_i)$.

By analogy with the case of Riemann integrals, we call (3.1) a composite quadrature formula for Riemann-Stieltjes integral.

Theorem 1. *Let AB be a smooth Jordan curve on the plane $\mathbb{R}^2$ and let $f,u:AB\to\mathbb{R}$ be Lipschitzian. Then*

$$|I(u,f)-I_n(u,f)|\leq\frac{3}{4}L_fL_u\sum_{i=1}^{n}S_i^2. \tag{3.2}$$

Proof. Replacing the integral I by

$$\int\limits_{AB} u(x,y)df(x,y)=\sum_{i=1}^{n}\int\limits_{A_{i-1}A_i} u(x,y)df(x,y)$$

we have

$$\begin{aligned}&|I(u,f)-I_n(u,f)|\\ =&\left|\sum_{i=1}^{n}\left[\int\limits_{A_{i-1}A_i} u(x,y)df(x,y)-\left(f(x_i,y_i)-f(x_{i-1},y_{i-1})\right)u(\xi_i,\eta_i)\right]\right|\\ \leq&\sum_{i=1}^{n}\left|\int\limits_{A_{i-1}A_i} u(x,y)df(x,y)-\left(f(x_i,y_i)-f(x_{i-1},y_{i-1})\right)u(\xi_i,\eta_i)\right|.\end{aligned}$$

Evaluating every term of the sum using Lemma 3, we have

$$\begin{aligned}&\left|\int\limits_{A_{i-1}A_i} u(x,y)df(x,y)-\left(f(x_i,y_i)-f(x_{i-1},y_{i-1})\right)u(\xi_i,\eta_i)\right|\\ \leq&\frac{3L_fL_u}{4}S_i^2,\end{aligned} \tag{3.3}$$

and then

$$|I(u,f) - I_n(u,f)| \le \frac{3L_f L_u}{4} \sum_{i=1}^{n} S_i^2,$$

or

$$\left| \int\limits_{AB} u(x,y) df(x,y) - \sum_{i=1}^{n} \left[f(x_i,y_i) - f(x_{i-1},y_{i-1}) \right] u(\xi_i,\eta_i) \right| \le \frac{3L_f L_u}{4} \sum_{i=1}^{n} S_i^2.$$

Hence the theorem is proved. ∎

Remark 1. *The constant in (3.2) can be decreased if we use Lemma 2 instead of Lemma 3. In this case, we have*

$$\begin{aligned} (3.4) \qquad & |I(u,f) - I_n(u,f)| \\ \le\ & \left| \int\limits_{AB} u(x,y) df(x,y) - \sum_{i=1}^{n} \frac{f(x_i,y_i) - f(x_{i-1},y_{i-1})}{S_i} \int\limits_{A_{i-1}A_i} u(x,y) ds \right| \\ \le\ & \frac{L_f L_u}{2} \sum_{i=1}^{n} S_i^2. \end{aligned}$$

Remark 2. *Since the fields with the larger values of L_u where $u = \dfrac{\varphi(x,y)}{\varphi^2(x,y) + \psi^2(x,y)}$ are frequently dealt with, it is helpful to replace in (3.2), L_u by the one that is considerably less then L_u. Sometimes it is possible (it is clear from the end of the proof of Lemma 3) if we write the right hand side of (2.10) as*

$$\frac{L_f}{2}(M-m)S + \frac{L_f L_u}{4} S^2.$$

4. Application to the Index Problem

Consider the integral (1.2), taking AB as the unit circle C

$$\begin{cases} x = \cos t \\ y = \sin t \\ 0 \le t \le 2\pi \end{cases},$$

$$(4.1) \qquad I \equiv \frac{1}{2\pi} \oint\limits_{x^2+y^2=1} \frac{\varphi(x,y) d\psi(x,y) - \psi(x,y) d\varphi(x,y)}{\varphi^2(x,y) + \psi^2(x,y)}.$$

Integral (4.1) defines the winding number (index) of the vector field

$$\mathbf{F}(x,y) = [\varphi(x,y), \psi(x,y)],$$

Since the winding number is an integer, we can evaluate integral (4.1) with an accuracy less than $\frac{1}{2}$.

Consider this integral as the sum of two integrals:

$$I = I_1 + I_2,$$

where

$$I_1 = \frac{1}{2\pi} \oint\limits_C \frac{\varphi(x,y)}{\varphi^2(x,y)+\psi^2(x,y)} d\psi(x,y)$$

and

$$I_2 = \frac{1}{2\pi} \oint\limits_C \frac{-\psi(x,y)}{\varphi^2(x,y)+\psi^2(x,y)} d\varphi(x,y).$$

If we compute each of these integrals with an accuracy less than 0.25, the resulting precision will be less than 0.5.

Let us subdivide the circle C by n equal parts. We denote the points of such subdivision by t_k $(k = 0, 1, 2, ..., n)$. The midpoint of the interval $[t_{k-1}, t_k]$ corresponds to the value of parameter $\tau_k = \frac{\pi(2k-1)}{n}$. Thus we have

$$\begin{aligned} t_k &= \frac{2\pi k}{n}, \\ t_{k-1} &= \frac{2\pi(k-1)}{n} \quad \text{and} \\ \tau_k &= \frac{\pi(2k-1)}{n}. \end{aligned}$$

The length of the arc S_k is equal to $\frac{2\pi}{n}$ so that

$$\sum_{i=1}^{n} S_i^2 = \sum_{i=1}^{n} \left(\frac{2\pi}{n}\right)^2 = \frac{4\pi^2}{n}.$$

Thus, the error evaluation (3.4) can be written as follows:

$$|I(u,f) - I_n(u,f)| \le \frac{3L_f L_u \pi^2}{n}. \tag{4.2}$$

The assumptions about the accuracy 0.25 of computation allow us to evaluate a number of terms in the quadrature formula (4.2). The number n can be obtained from the inequality

$$\frac{3L_f L_u \pi^2}{n} < \frac{1}{4} \quad \text{implies} \quad n > 12L_f L_u \pi^2.$$

Taking into account the presence of the factor $\frac{1}{2\pi}$ in the Poincaré formula, we can decrease $12L_f L_u \pi^2$ to the value $6L_f L_u \pi$, but in this case, we use formula (4.2) and set (for example in integral I_1)

$$u(x,y) = \frac{\varphi(x,y)}{\varphi^2(x,y)+\psi^2(x,y)}, \quad \psi(x,y) = \psi(x,y).$$

The number n must satisfy the condition

$$n > 6L_f L_u \pi.$$

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