

SOME INEQUALITIES FOR RANDOM VARIABLES WHOSE PROBABILITY DENSITY FUNCTIONS ARE ABSOLUTELY CONTINUOUS USING A PRE-CHEBYCHEV INEQUALITY

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ABSTRACT. Using the pre-Chebychev inequality considered by Matić, Pečarić and Ujević in [2], some inequalities are obtained for random variables whose p.d.f.s are absolutely continuous and whose derivatives are in $L_\infty [a, b]$.

1. INTRODUCTION

The following inequality is well known in the literature as Chebychev's inequality (see for example [1, p. 297]).

Theorem 1. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two absolutely continuous mappings on $[a, b]$ whose derivatives $f', g' : [a, b] \rightarrow \mathbb{R}$ belong to the Lebesgue space $L_\infty [a, b]$. Then,*

$$(1.1) \quad \left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx \right| \leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty.$$

The constant $\frac{1}{12}$ is the best possible.

In [2], Matić, Pečarić and Ujević proved the following refinement of (1.1) which we call the "pre-Chebychev" inequality

$$(1.2) \quad \left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx \right| \leq \frac{1}{2\sqrt{3}} (b-a) \|f'\|_\infty \left[\frac{1}{b-a} \int_a^b g^2(x) dx - \left(\frac{1}{b-a} \int_a^b g(x) dx \right)^2 \right]^{\frac{1}{2}},$$

provided that f is as in Theorem 1 and all the integrals in (1.2) exist and are finite.

Matić, Pečarić and Ujević observed that: if a factor is known, say $g(t)$, $t \in [a, b]$, then instead of using (1.1) to estimate the difference

$$\frac{1}{b-a} \int_a^b f(t) g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt,$$

it is better to use (1.2). They demonstrated this by improving some results of the second author in [6] related to Taylor's formula with integral remainder.

Date: January 18, 2000.

1991 Mathematics Subject Classification. Primary 26D15; Secondary 65Xxx.

Key words and phrases. Pre-Chebychev Inequality, Random Variables, Expectation, Cumulative Distribution Function.

Using the same approach here, we obtain some inequalities for the expectation, $E(X)$, and cumulative distribution function $F(x)$ of a random variable having the probability density function $f : [a, b] \rightarrow \mathbb{R}$ which is assumed to be absolutely continuous and whose derivative $f' \in L_\infty [a, b]$.

2. SOME INEQUALITIES

We start with the following result for expectation.

Theorem 2. *Let X be a random variable having the probability density function $f : [a, b] \rightarrow \mathbb{R}$. Assume that f is absolutely continuous on $[a, b]$ and $f' \in L_\infty [a, b]$. Then,*

$$(2.1) \quad \left| E(X) - \frac{a+b}{2} \right| \leq \frac{1}{12} (b-a)^2 \|f'\|_\infty,$$

where $E(X)$ is the expectation of the random variable X .

Proof. If we put $g(t) = t$ in (1.2),

$$(2.2) \quad \left| \frac{1}{b-a} \int_a^b t f(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b t dt \right| \\ \leq \frac{1}{2\sqrt{3}} (b-a) \|f'\|_\infty \left[\frac{1}{b-a} \int_a^b t^2 dt - \left(\frac{1}{b-a} \int_a^b t dt \right)^2 \right]^{\frac{1}{2}}.$$

However,

$$\frac{1}{b-a} \int_a^b t^2 dt - \left(\frac{1}{b-a} \int_a^b t dt \right)^2 = \frac{(b-a)^2}{12}$$

and so (2.1) is true. ■

Remark 1. *We could obtain the same inequality by applying Chebychev's inequality (1.1). Note, however, that for further results, the pre-Chebychev inequality provides a better estimate than would be obtained using the classical result (1.1).*

Theorem 3. *Let X and f be as above. If*

$$\sigma_\mu(X) := \left[\int_a^b (t-\mu)^2 f(t) dt \right]^{\frac{1}{2}}, \quad \mu \in [a, b],$$

then,

$$(2.3) \quad \left| \sigma_\mu^2(X) - \left(\mu - \frac{a+b}{2} \right)^2 - \frac{1}{12} (b-a)^2 \right| \\ \leq \frac{1}{2\sqrt{3}} (b-a)^2 \left[\frac{1}{3} \left(\mu - \frac{a+b}{2} \right)^2 + \frac{1}{180} (b-a)^2 \right] \|f'\|_\infty \\ \leq \frac{1}{3\sqrt{15}} (b-a)^3 \|f'\|_\infty,$$

for all $\mu \in [a, b]$.

Proof. If $g(t) = (t - \mu)^2$ in (1.2), then,

$$(2.4) \quad \left| \frac{1}{b-a} \int_a^b (t - \mu)^2 f(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b (t - \mu)^2 dt \right| \\ \leq \frac{1}{2\sqrt{3}} \|f'\|_\infty \left[\frac{1}{b-a} \int_a^b (t - \mu)^4 dt - \left(\frac{1}{b-a} \int_a^b (t - \mu)^2 dt \right)^2 \right]^{\frac{1}{2}}.$$

However,

$$\frac{1}{b-a} \int_a^b (t - \mu)^2 dt = \left(\mu - \frac{a+b}{2} \right)^2 + \frac{1}{12} (b-a)^2$$

and

$$\begin{aligned} & \frac{1}{b-a} \int_a^b (t - \mu)^4 dt - \left(\frac{1}{b-a} \int_a^b (t - \mu)^2 dt \right)^2 \\ &= \frac{1}{5} \cdot \frac{(b - \mu)^5 + (\mu - a)^5}{b - a} - \left[\frac{(b - \mu)^3 + (\mu - a)^3}{3(b - a)} \right]^2 \\ &= \frac{1}{45} \left[4 \left[(b - \mu)^2 - (\mu - a)^2 \right]^2 + 2(b - \mu)^2 (\mu - a)^2 \right. \\ & \quad \left. + (\mu - a)(b - \mu) \left[(b - \mu)^2 + (\mu - a)^2 \right] \right] := A, \end{aligned}$$

which simplifies further to give:-

$$\begin{aligned} A &= \frac{(b-a)^2}{45} \left[15 \left(\mu - \frac{a+b}{2} \right)^2 + \frac{1}{4} (b-a)^2 \right] \\ &= (b-a)^2 \left[\frac{1}{3} \left(\mu - \frac{a+b}{2} \right)^2 + \frac{1}{180} (b-a)^2 \right]. \end{aligned}$$

Using (2.4), we deduce the desired inequality (2.3). ■

The best inequality we can obtain from (2.3) is that for which $\mu = \frac{a+b}{2}$, giving the following corollary.

Corollary 1. *With the above assumptions and denoting $\sigma_0(X) := \sigma_{\frac{a+b}{2}}(X)$,*

$$(2.5) \quad \left| \sigma_0^2(X) - \frac{(b-a)^2}{12} \right| \leq \frac{1}{12\sqrt{15}} (b-a)^3 \|f'\|_\infty.$$

The following theorem provides an inequality that connects the expectation $E(X)$ and the cumulative distribution function $F(x) := \int_a^x f(t) dt$ of a random variable X having the p.d.f. $f : [a, b] \rightarrow \mathbb{R}$.

Theorem 4. *Let X be a random variable whose p.d.f., $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$ and $f' \in L_\infty[a, b]$. Then,*

$$(2.6) \quad \left| E(X) + (b-a)F(x) - x - \frac{b-a}{2} \right| \leq \frac{1}{12} (b-a)^3 \|f'\|_\infty$$

for all $x \in [a, b]$.

Proof. We use the following equality established by Barnett and Dragomir in [4]

$$(2.7) \quad (b-a)F(x) + E(X) - b = \int_a^b p(x,t) dF(t) = \int_a^b p(x,t) f(t) dt,$$

where

$$p(x,t) := \begin{cases} t-a & \text{if } a \leq t \leq x \leq b \\ t-b & \text{if } a \leq x < t \leq b \end{cases}.$$

Now, if we apply the inequality (1.2) for $g(t) = p(x,t)$, we obtain

$$(2.8) \quad \left| \frac{1}{b-a} \int_a^b p(x,t) f(t) dt - \frac{1}{b-a} \int_a^b p(x,t) dt \cdot \frac{1}{b-a} \int_a^b f(t) dt \right| \\ \leq \frac{1}{2\sqrt{3}} (b-a) \|f'\|_\infty \left[\frac{1}{b-a} \int_a^b p^2(x,t) dt - \left(\frac{1}{b-a} \int_a^b p(x,t) dt \right)^2 \right]^{\frac{1}{2}}.$$

Observe that

$$\frac{1}{b-a} \int_a^b p(x,t) dt = x - \frac{a+b}{2},$$

and

$$\begin{aligned} D & : = \frac{1}{b-a} \int_a^b p^2(x,t) dt - \left(\frac{1}{b-a} \int_a^b p(x,t) dt \right)^2 \\ & = \frac{1}{b-a} \left[\frac{(b-x)^3 + (x-a)^3}{3} \right] - \left(x - \frac{a+b}{2} \right)^2 \\ & = \frac{1}{12} (b-a)^2. \end{aligned}$$

Using (2.8), we deduce (2.6). ■

Remark 2. If in (2.6) either $x = a$ or $x = b$,

$$\left| E(X) - \frac{a+b}{2} \right| \leq \frac{1}{12} (b-a)^3 \|f'\|_\infty,$$

which is inequality (2.1).

Remark 3. If in (2.6) $x = \frac{a+b}{2}$, then

$$(2.9) \quad \left| E(X) + (b-a) \Pr \left(X \leq \frac{a+b}{2} \right) - b \right| \leq \frac{1}{12} (b-a)^3 \|f'\|_\infty.$$

Theorem 5. Let X , F and f be as above. Then,

$$(2.10) \quad \left| E(X) + \frac{b-a}{2} F(x) - \frac{x+b}{2} \right| \\ \leq \frac{1}{4} (b-a) \|f'\|_\infty \left[\left(x - \frac{a+b}{2} \right)^2 + \frac{1}{12} (b-a)^2 \right] \\ \leq \frac{1}{12} (b-a)^3 \|f'\|_\infty$$

for all $x \in [a, b]$.

Proof. Using the same identity of Barnett and Dragomir [4] as in Theorem 4 and applying the pre-Chebyshev inequality (1.2), for $x \in [a, b]$ we get:-

$$\begin{aligned}
(2.11) \quad & \left| \frac{1}{x-a} \int_a^x (t-a) f(t) dt - \frac{1}{x-a} \int_a^x (t-a) dt \cdot \frac{1}{x-a} \int_a^x f(t) dt \right| \\
& \leq \frac{1}{2\sqrt{3}} (x-a) \|f'\|_\infty \left[\frac{1}{x-a} \int_a^x (t-a)^2 dt - \left(\frac{1}{x-a} \int_a^x (t-a) dt \right)^2 \right]^{\frac{1}{2}} \\
& = \frac{1}{12} (x-a)^2 \|f'\|_\infty
\end{aligned}$$

and, similarly,

$$\begin{aligned}
(2.12) \quad & \left| \frac{1}{b-x} \int_x^b (t-b) f(t) dt - \frac{1}{b-x} \int_x^b (t-b) dt \cdot \frac{1}{b-x} \int_x^b f(t) dt \right| \\
& \leq \frac{1}{12} (b-x)^2 \|f'\|_\infty,
\end{aligned}$$

for all $x \in [a, b]$.

From (2.11) and (2.12) we can write

$$(2.13) \quad \left| \int_a^x (t-a) f(t) dt - \frac{x-a}{2} F(x) \right| \leq \frac{1}{12} (x-a)^3 \|f'\|_\infty$$

and

$$(2.14) \quad \left| \int_x^b (t-b) f(t) dt + \frac{b-x}{2} (1-F(x)) \right| \leq \frac{1}{12} (b-x)^3 \|f'\|_\infty,$$

for all $x \in [a, b]$.

Summing (2.13) and (2.14) and using the triangle inequality, we deduce

$$\begin{aligned}
& \left| \int_a^x (t-a) f(t) dt + \int_x^b (t-b) f(t) dt - \frac{b-a}{2} F(x) + \frac{b-x}{2} \right| \\
& \leq \frac{1}{12} \|f'\|_\infty \left[(x-a)^3 + (b-x)^3 \right] \\
& = \frac{1}{12} (b-a) \|f'\|_\infty \left[3 \left(x - \frac{a+b}{2} \right)^2 + \frac{1}{4} (b-a)^2 \right] \\
& = \frac{1}{4} (b-a) \|f'\|_\infty \left[\left(x - \frac{a+b}{2} \right)^2 + \frac{1}{12} (b-a)^2 \right].
\end{aligned}$$

Using the identity (2.7), the desired result (2.10) is obtained. ■

Remark 4. If in (2.10) either $x = a$ or $x = b$, the inequality (2.1) is recaptured.

Remark 5. If in (2.10), $x = \frac{a+b}{2}$, then the best inequality that can be obtained is:-

$$\left| E(X) + \frac{b-a}{2} \Pr \left(X \leq \frac{a+b}{2} \right) - \frac{a+3b}{4} \right| \leq \frac{1}{48} (b-a)^3 \|f'\|_\infty.$$

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