Weighted Hardy inequalities with mixed norms I [1]

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Abstract

We obtain in this paper conditions on the nonnegative weight functions u(x) and v(x) which ensure an inequality of the form

$$\left(\int_{-\infty}^{\infty} \mid (Tf)(x)u(x) \mid^{(1-\frac{1}{s})q} dx\right)^{\frac{1}{(1-\frac{1}{s})q}} \leq AC^{(1-\frac{1}{s})} \left(\int_{-\infty}^{\infty} \mid f(x)v(x) \mid^{(1-\frac{1}{r})p} dx\right)^{\frac{1}{(1-\frac{1}{r})p}},$$

where T is either K or K^*, A and C are constants depending on (k, p, q, r, s) but independent of f.

Furthermore, conditions on the nonnegative weight functions u(x) and v(x) for which the above inequality is reversed is also obtained.

Some consequences of our results are also given.

1 Introduction

Let $k(x, y) \geq 0$ be defined on $\triangle = \{(x, y) \in \Re^2 : y < x\}$ and define the operator K and its dual K^* by

$$(Kf)(x) = \int_{-\infty}^{x} k(x, y)f(y)dy, \quad (K^*f)(x) = \int_{x}^{\infty} k(y, x)f(y)dy \tag{1.1}$$

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We shall in this paper give conditions on the nonnegative weight functions u(x) and v(x) in terms of the kernel k(x,y) and the indices p,q,r,s for which the inequality

$$\left(\int_{-\infty}^{\infty} \mid (Tf)(x)u(x) \mid^{(1-\frac{1}{s})q} dx\right)^{\frac{1}{(1-\frac{1}{s})q}} \leq AC^{(1-\frac{1}{s})} \left(\int_{-\infty}^{\infty} \mid f(x)v(x) \mid^{(1-\frac{1}{r})p} dx\right)^{\frac{1}{(1-\frac{1}{r})p}} \quad (1.2)$$

holds, where T is either K or K^*, and C is constant independent of f. The specific case of the Hardy's operator with the kernel $k(x,y) \equiv 1$ was considered by Bradley[3] for the case $p, q \geq 1$, Beesack and Heinig[2] considered it for the case $p, q < 1$,

while Anderson and Heinig[1] considered it for a general class of kernels k(x,y).

In this paper, we shall study the Hardy's operator for a more general kernel k(x,y) with some restrictions placed on the parameters p,q,r,s,p' with the intention to obtain some results which will compliments the earlier ones obtained in [1].

Throughout this paper, we shall let p' denote the conjugate index of p and is defined by $\frac{1}{p} + \frac{1}{p'} = 1 - \frac{1}{r},\ \ r > 0 \text{ and } r \neq 1$. The conjugate q' of q is similarly defined by $\frac{1}{q} + \frac{1}{q'} = 1 - \frac{1}{s}$, $s > 0 \text{ and } s \neq 1$. It is clear from this that p' may be negative.

Furthermore, we shall assume that $1 < p \leq q < \infty,\ \ 1 < (1-\frac{1}{r})p \leq (1-\frac{1}{s})q$. From this it follows that

$$\frac{(1-\frac{1}{s})q}{(1-\frac{1}{r})p} \geq 1.$$

A and C denote constants which may be different at difference occurences. Unless otherwise stated, we shall define a nonnegative function h(y) by

$$h(y) = \left(\int_{-\infty}^{y} k(y,z)^{(1-\beta)(1-\frac{1}{r})p'} v(z)^{-(1-\frac{1}{r})p'} dz\right)^{-\frac{1}{(1-\frac{1}{r})p'+(1-\frac{1}{s})q}} \quad (1.3)$$

We also observe that $(1-\frac{1}{r})p' + (1-\frac{1}{s})q$ may be greater than zero or less than zero. This will be studied in sections 2 and 3 respectively.

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In this section, we shall assume that $(1-\frac{1}{r})p' + (1-\frac{1}{s})q > 0$, where $p' = \frac{p}{(1-\frac{1}{r})p-1} > 0$.

Before stating our main results in this section, we shall need the following definition and lemmas.

Definition 2.1 *Let $u(x) \geq 0, v(x) \geq 0, 1 < p \leq q < \infty$, and suppose $k(x,y) \geq 0$ is defined in $\triangle = \{(x,y) \in \Re^2 : y < x\}$. (a) We shall say that the pair of functions (u(x),v(x)) satisfies the A(k,p,q,r,s) condition with a constant C if their exists a real number β, $0 \leq \beta \leq 1$ such that for all a*

$$\left(\int_a^\infty k(y,a)^{(1-1/s)\beta q} u(y)^{(1-1/s)q} dy\right)^{\frac{1}{(1-1/s)q}} \cdot \left(\int_{-\infty}^a K(a,y)^{(1-1/r)(1-\beta)p'} v(y)^{-(1-1/r)p'} dy\right)^{\frac{1}{(1-1/r)p'}} \leq C < \infty, \tag{2.4}$$

holds.

(b) We shall say that the pair of functions (u(x),v(x)) satisfies the $A^*(k,p,q,r,s)$ condition with a constant C^* if their exists a real number β, $0 \leq \beta \leq 1$ such that for all a

$$\left(\int_{-\infty}^a K(a,y)^{(1-1/s)\beta q} u(y)^{(1-1/s)q} dy\right)^{\frac{1}{(1-1/s)q}} \cdot \left(\int_a^\infty K(y,a)^{(1-1/r)(1-\beta)p'} v(y)^{-(1-1/r)p'} dy\right)^{\frac{1}{(1-1/r)p'}} \leq C^* < \infty, \tag{2.5}$$

holds.

Lemma 2.1 *Let $f(x,y) \geq 0$.Suppose $1 \leq p \leq \infty$ and $b > -\infty$, then*

$$\left(\int_b^\infty \left[\int_b^x f(x,y)dy\right]^p dx\right)^{1/p} \leq \int_b^\infty \left(\int_y^\infty f(x,y)^p dx\right)^{1/p} dy \tag{2.6}$$

and

$$\left(\int_b^\infty \left[\int_x^\infty f(x,y)dy\right]^p dx\right)^{1/p} \leq \int_b^\infty \left(\int_b^y f(x,y)^p dx\right)^{1/p} dy \tag{2.7}$$

Proof: See [4, Theorem 202, p. 148].

Lemma 2.2 *For any $f \geq 0$ and any $\alpha > 0$ we have*

$$\int_a^b f(x) \left(\int_a^x f(t)dt\right)^\alpha dx = \frac{1}{(\alpha+1)} \left(\int_a^b f(x)dx\right)^{(\alpha+1)} \tag{2.8}$$

Proof: Let

$$F(x) = \int_a^x f(t)dt$$

Then

$$F'(x) = f(x)dx$$

Therefore

$$\begin{aligned}\int_a^b f(x)\left(\int_a^x f(t)dt\right)^\alpha dx &= \int_a^b F(x)^\alpha dF(x)\\ &= \frac{1}{(\alpha+1)}\left(\int_a^b f(x)dx\right)^{(\alpha+1)}\end{aligned}$$

Hence

$$\int_a^b f(x)\left(\int_a^x f(t)dt\right)^\alpha dx = \frac{1}{(\alpha+1)}\left(\int_a^b f(x)dx\right)^{(\alpha+1)}$$

Lemma 2.3 *Let $k(x,y) \geq 0$ be defined in $\triangle = \{(x,y) \in \Re^2 : y < x\}$ with k(x,y) nondecreasing in y and nonincreasing in x. Then the following inequality holds*

$$\begin{aligned}&\textstyle\int_{-\infty}^x k(x,y)^{(1-\beta)(1-1/r)p'} v(y)^{-(1-1/r)p'} h(y)^{-(1-1/r)p'} dy\\ &\qquad\leq \frac{(1-1/r)p' + (1-1/s)q}{(1-1/s)q} h(x)^{(1-1/s)q}\end{aligned} \tag{2.9}$$

Proof: For any $x \in \Re$, let

$$\begin{aligned}J &= \int_{-\infty}^x k(x,y)^{(1-\beta)(1-1/r)p'} v(y)^{-(1-1/r)p'} h(y)^{-(1-1/r)p'} dy\\ &= \int_{-\infty}^x k(x,y)^{(1-\beta)(1-1/r)p'} v(y)^{-(1-1/r)p'}\\ &\quad\cdot\left(\int_{-\infty}^y k(y,z)^{(1-\beta)(1-1/r)p'} v(z)^{-(1-1/r)p'} dz\right)^{-\frac{(1-1/r)p'}{(1-1/r)p'+(1-1/s)q}} dy \quad \text{by } (1.3).\end{aligned}$$

Since $y < x$, it follows that $k(y,z) \geq k(x,z)$ and this is bounded above by

$$\begin{aligned} J &\leq \int_{-\infty}^{x} k(x,y)^{(1-\beta)(1-1/r)p'} v(y)^{-(1-1/r)p'} \\ &\cdot \left(\int_{-\infty}^{y} k(x,z)^{(1-\beta)(1-1/r)p'} v(z)^{-(1-1/r)p'} dz\right)^{-\frac{(1-1/r)p'}{(1-1/r)p'+(1-1/s)q}} dy \\ &= \frac{(1-1/r)p' + (1-1/s)q}{(1-1/s)q} \left(\int_{-\infty}^{x} k(x,z)^{(1-\beta)(1-1/r)p'} v(z)^{-(1-1/r)p'} dz\right)^{\frac{(1-1/s)q}{(1-1/r)p'+(1-1/s)q}} \\ &\quad \text{by Lemma2.2} \\ &= \frac{(1-1/r)p' + (1-1/s)q}{(1-1/s)q} h(x)^{(1-1/s)q} \end{aligned}$$

This completes the proof of the Lemma.

Lemma 2.4 *Let $k(x,y) \geq 0$ be defined in $\triangle = \{(x,y) \in \Re^2 : y < x\}$ with k(x,y) nondecreasing in y and nonincreasing in x. Suppose (u(x),v(x)) satisfies the A(k,p,q,r,s) condition with constant C and if $(1-\frac{1}{r})^2 pp' - (1-\frac{1}{s})q^2 = 0$. Then*

$$\left(\int_y^\infty k(x,y)^{(1-1/s)\beta q} u(x)^{(1-1/s)q} h(x)^{(1-1/s)(1-1/r)p} dx\right)^{\frac{(1-1/s)q}{(1-1/r)p'}} \leq \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/r)p'}\right)^{\frac{(1-1/s)q}{(1-1/r)p'}}$$

$$\cdot C^{\frac{(1-1/s)^2(1-1/r)pq}{(1-1/r)p'+(1-1/s)q}} \left(\int_{-\infty}^{y} k(z,y)^{(1-1/s)\beta q} u(z)^{(1-1/s)q} dz\right)^{\frac{(1-1/s)q}{(1-1/r)p'+(1-1/s)q}} \qquad (2.10)$$

Proof: Let

$$\begin{aligned} J &= \int_y^\infty k(x,y)^{(1-1/s)\beta q} u(x)^{(1-1/s)q} h(x)^{(1-1/s)(1-1/r)p} dx \\ &= \int_y^\infty k(x,y)^{(1-1/s)\beta q} u(x)^{(1-1/s)q} \\ &\quad \cdot \left(\int_{-\infty}^{x} k(x,z)^{(1-\beta)(1-1/r)p'} v(z)^{-(1-1/r)p'} dz\right)^{\frac{(1-1/s)(1-1/r)p}{(1-1/r)p'+(1-1/s)q}} dy \quad 1.3 \end{aligned} \qquad (2.11)$$

Since $(u(x), v(x))$ satisfies the A(k,p,q,r,s) condition with constant C, we have obtain from (2.4) that

$$\int_{-\infty}^{x} k(x,z)^{(1-\beta)(1-1/r)p'} v(z)^{-(1-1/r)p'} dz$$

$$\leq \quad C^{(1-1/r)p'} \left(\int_{x}^{\infty} k(z,x)^{(1-1/s)\beta q} u(z)^{(1-1/s)q} dz\right)^{\frac{-(1-1/r)p'}{(1-1/s)q}} \tag{2.12}$$

Substitute (2.12) into (2.11) and apply Lemma2.3 we have

$$\begin{aligned}
J \quad &\leq \quad C^{\frac{(1-1/s)(1-1/r)pp'}{(1-1/r)p'+(1-1/s)q}} \int_{y}^{\infty} k(x,y)^{(1-1/s)\beta q} u(x)^{(1-1/s)q} \\
&\cdot \left(\int_{x}^{\infty} k(z,x)^{(1-1/s)\beta q} u(z)^{(1-1/s)q} dz\right)^{\frac{-(1-1/s)q}{(1-1/r)p'+(1-1/s)q}} dx \\
&= \quad \frac{(1-1/r)p'+(1-1/s)q}{(1-1/r)} C^{\frac{(1-1/s)(1-1/r)^2 pp'}{(1-1/r)p'+(1-1/s)q}} \\
&\cdot \int_{y}^{\infty} k(x,y)^{(1-1/s)\beta q} u(x)^{(1-1/s)q} \left(\int_{-\infty}^{y} k(z,y)^{(1-1/s)\beta q} u(z)^{(1-1/s)q} dz\right)^{\frac{(1-1/r)p'}{(1-1/r)p'+(1-1/s)q}}
\end{aligned}$$

Hence

$$\begin{aligned}
J^{\frac{(1-1/s)q}{(1-1/r)p'}} \quad &\leq \quad \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/r)p'}\right)^{\frac{(1-1/s)q}{(1-1/r)p'}} C^{\frac{(1-1/s)^2(1-1/r)pq}{(1-1/r)p'+(1-1/s)q}} \\
&\cdot \left(\int_{-\infty}^{y} k(z,y)^{(1-1/s)\beta q} u(z)^{(1-1/s)q} dz\right)^{\frac{(1-1/s)q}{(1-1/r)p'+(1-1/s)q}}
\end{aligned}$$

and the proof is complete.

Theorem 2.1 *Let K be an integral operator defined in (1.1). Let $k(x,y) \geq 0$ is defined in $\triangle = \{(x,y) \in \Re^2 : y < x\}$ with k(x,y) nonincreasing in x and nondecreasing in y. If (u(x),v(x)) satisfies the A(k,p,q,r,s) condition with constant C and $(1-\frac{1}{r})^2 pp' - (1-\frac{1}{s})q^2 = 0$. Then*

$$\left(\int_{-\infty}^{\infty} \mid (Tf)(x)u(x) \mid^{(1-\frac{1}{s})q} dx\right)^{\frac{1}{(1-\frac{1}{s})q}} \quad \leq \quad AC^{(1-\frac{1}{s})} \left(\int_{-\infty}^{\infty} \mid f(x)v(x) \mid^{(1-\frac{1}{r})p} dx\right)^{\frac{1}{(1-\frac{1}{r})p}} \tag{2.13}$$

where

$$A \quad = \quad \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)q}\right)^{\frac{1}{(1-1/r)p'}} \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/r)p'}\right)^{\frac{1}{q}}$$

Proof: Assume $f \geq 0$ for which the right hand side of (2.13) is finite and let

$$\begin{aligned} J &= \int_{-\infty}^{\infty} \mid u(x)(Tf)(x) \mid^{(1-1/s)q} dx \\ &= \int_{-\infty}^{\infty} u(x)^{(1-1/s)q} \left(\int_{-\infty}^{x} k(x,y)f(y)dy \right)^{(1-1/s)q} dx \\ &= \int_{-\infty}^{\infty} u(x)^{(1-1/s)q} \left(\int_{-\infty}^{x} k(x,y)^{\beta} f(y)v(y)h(y)k(x,y)^{1-\beta} v(y)^{-1} h(y)^{-1} dy \right)^{(1-1/s)q} dx \end{aligned}$$

By Holder's inequality, we have

$$J \leq \int_{-\infty}^{\infty} u(x)^{(1-1/s)q} \left(\int_{-\infty}^{x} k(x,y)^{(1-1/r)\beta p} [f(y)v(y)h(y)]^{(1-1/r)p} dy \right)^{\frac{(1-1/s)q}{(1-1/r)p}}$$

$$\cdot \left(\int_{-\infty}^{x} k(x,y)^{(1-1/r)(1-\beta)p'} [v(y)h(y)]^{-(1-1/r)p'} dy \right)^{\frac{(1-1/s)q}{(1-1/r)p'}} dx$$

By Lemma2.3, we obtain

$$\begin{aligned} J &\leq \left(\frac{(1-1/r)p' + (1-1/s)q}{(1-1/s)q} \right)^{\frac{(1-1/s)q}{(1-1/r)p'}} \int_{-\infty}^{\infty} u(x)^{(1-1/s)q} \\ &\quad \cdot \left(\int_{-\infty}^{x} k(x,y)^{(1-1/r)\beta p} [f(y)v(y)h(y)]^{(1-1/r)p} dy \right)^{\frac{(1-1/s)q}{(1-1/r)p}} h(x)^{(1-1/s)(1-1/r)p} \end{aligned}$$

By Minkowskii's integral inequality (2.6), we obtain

$$\begin{aligned} J &\leq \left(\frac{(1-1/r)p' + (1-1/s)q}{(1-1/s)q} \right)^{\frac{(1-1/s)q}{(1-1/r)p'}} \left(\int_{-\infty}^{\infty} \left\{ \int_{y}^{\infty} k(x,y)^{(1-1/s)\beta q} u(x)^{(1-1/s)q} \right.\right. \\ &\quad \left.\left. \cdot\ h(x)^{(1-1/r)(1-1/s)p} dx \right\}^{\frac{(1-1/r)p}{(1-1/s)q}} [f(y)v(y)h(y)]^{(1-1/r)p} dy \right)^{\frac{(1-1/s)q}{(1-1/r)p}} \end{aligned}$$

By Lemma2.4, we obtain

$$J \leq \left(\frac{(1-1/r)p' + (1-1/s)q}{(1-1/s)q} \right)^{\frac{(1-1/s)q}{(1-1/r)p'}} \left(\frac{(1-1/r)p' + (1-1/s)q}{(1-1/s)p'} \right)^{(1-1/s)} C^{\frac{(1-1/s)^2(1-1/r)^2 pp'}{(1-1/r)p' + (1-1/s)q}}$$

$$\cdot \left(\int_{\infty}^{\infty} [f(y)v(y)h(y)]^{(1-1/r)p} \left(\int_y^{\infty} k(y,z)^{(1-1/s)\beta q} \right.\right.$$
$$\left.\left. \cdot u(z)^{(1-1/s)q} dz\right)^{\frac{(1-1/s)q}{(1-1/r)p'+(1-1/s)q}} dy\right)^{\frac{(1-1/s)q}{(1-1/r)p}} \tag{2.14}$$

Since (u(x),v(x)) satisfies the A(k,p,q,r,s) condition with constant C, we have from (2.4) that

$$\left(\int_y^{\infty} k(y,z)^{(1-1/s)\beta q} u(z)^{(1-1/s)q} dz\right)^{\frac{(1-1/s)q}{(1-1/r)p'+(1-1/s)q}}$$

$$\leq \; C^{\frac{(1-1/s)(1-1/r)^2 pp'}{(1-1/r)p'+(1-1/s)q}} \left(\int_y^{\infty} k(y,z)^{(1-1/r)(1-\beta)p'} v(z)^{-(1-1/r)p'} dz\right)^{-\frac{(1-1/r)p}{(1-1/r)p'+(1-1/s)q}} \tag{2.15}$$

Substitute (2.15) into (2.14) we obtain

$$J \;\leq\; \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)q}\right)^{\frac{(1-1/s)q}{(1-1/r)p'}} \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)p'}\right)^{(1-1/s)}$$
$$\cdot C^{\frac{(1-1/s)^2[(1-1/r)^2 pp'+(1-1/r)p'q]}{(1-1/r)p'+(1-1/s)q}} \left(\int_{\infty}^{\infty} [f(y)v(y)h(y)]^{(1-1/r)p} \right.$$
$$\left. \cdot \left(\int_y^{\infty} k(y,z)^{(1-1/r)(1-\beta)p'} v(z)^{-(1-1/r)p'} dz\right)^{-\frac{(1-1/r)p}{(1-1/r)p'+(1-1/s)q}} dy\right)^{\frac{(1-1/s)q}{(1-1/r)p}}$$

By (1.3) we have

$$J \;\leq\; \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)q}\right)^{\frac{(1-1/s)q}{(1-1/r)p'}} \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)p'}\right)^{(1-1/s)}$$
$$\cdot C^{\frac{(1-1/s)^2[(1-1/s)q^2+(1-1/r)p'q]}{(1-1/r)p'+(1-1/s)q}} \left(\int_{\infty}^{\infty} [f(y)v(y)h(y)]^{(1-1/r)p} h(y)^{(1-1/r)p} h(y)^{-(1-1/r)p} dy\right)^{\frac{(1-1/s)q}{(1-1/r)p}}$$

Hence

$$J^{\frac{1}{(1-1/s)q}} \;\leq\; \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)q}\right)^{\frac{1}{(1-1/r)p'}} \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)p'}\right)^{\frac{1}{q}}$$
$$\cdot C^{(1-1/s)} \left(\int_{\infty}^{\infty} [f(y)v(y)]^{(1-1/r)p} dy\right)^{(1-1/r)p}$$

This completes the proof of the theorem.

Remark 2.1 *In the limit $r \to \infty$ and $s \to \infty$ in in Theorem 2.1, we obtain*

$$\left(\int_{-\infty}^{\infty} \mid (Kf)(x)u(x) \mid^q dx\right)^{\frac{1}{q}} \leq AC\left(\int_{-\infty}^{\infty} \mid f(x)v(x) \mid^p dx\right)^{\frac{1}{p}} \tag{2.16}$$

where

$$A = \left(\frac{p'+q}{q}\right)^{\frac{1}{p'}}\left(\frac{p'+q}{p'}\right)^{\frac{1}{q}}$$

which was a recent result obtained by Anderson and Heining [1].

Corollary 2.1 *In Theorem 2.1, if we set $q^2 = 4(1-\frac{1}{s})$ and $p = p' = 2$, we obtain $r = s \to \infty$ and so (2.13) reduces to*

$$\left(\int_{-\infty}^{\infty} \mid (Tf)(x)u(x) \mid^2 dx\right)^{\frac{1}{2}} \leq C\left(\int_{-\infty}^{\infty} \mid f(x)v(x) \mid^2 dx\right)^{\frac{1}{2}} \tag{2.17}$$

Theorem 2.2 *Let K^* be an integral operator defined in (1). Let $k(x,y) \geq 0$ is defined in $\triangle = \{(x,y) \in \Re^2 : y < x\}$ with k(x,y) nonincreasing in x and nondecreasing in y. If (u(x),v(x)) satisfies the $A^*(k,p,q,r,s)$ condition with constant C^* and $(1-\frac{1}{r})^2pp' - (1-\frac{1}{s})q^2 = 0$. Then*

$$\left(\int_{-\infty}^{\infty} \mid (K^*f)(x)u(x) \mid^{(1-\frac{1}{s})q} dx\right)^{\frac{1}{(1-\frac{1}{s})q}} \leq AC^{*(1-\frac{1}{s})}\left(\int_{-\infty}^{\infty} \mid f(x)v(x) \mid^{(1-\frac{1}{r})p} dx\right)^{\frac{1}{(1-\frac{1}{r})p}} \tag{2.18}$$

where

$$A = \left(\frac{(1-1/r)p' + (1-1/s)q}{(1-1/s)q}\right)^{\frac{1}{(1-1/r)p'}}\left(\frac{(1-1/r)p' + (1-1/s)q}{(1-1/r)p'}\right)^{\frac{1}{q}}$$

Proof: The proof is similar to that of Theorem 2.1 except that we define h(y) by

$$h(y) = \left(\int_y^{\infty} K(y,z)^{(1-\beta)(1-1/r)p'} v(z)^{-(1-1/r)p'} dz\right)^{\frac{-1}{(1-1/r)p+(1-1/s)q}}$$

Remark 2.2 *If in Theorem 2.2 we let $r \to \infty$ and $s \to \infty$,then our result reduces to*

$$\left(\int_{-\infty}^{\infty} \mid (K^*f)(x)u(x) \mid^q dx\right)^{\frac{1}{q}} \leq AC^*\left(\int_{-\infty}^{\infty} \mid f(x)v(x) \mid^p dx\right)^{\frac{1}{p}} \tag{2.19}$$

where

$$A = \left(\frac{p'+q}{q}\right)^{\frac{1}{p'}}\left(\frac{p'+q}{p'}\right)^{\frac{1}{q}}$$

which is an estimate obtained by Andersen and Heinig[1].

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Throughout this section, we shall assume that $(1-\frac{1}{r})p' + (1-\frac{1}{s})q < 0$, $p' < 0$. $(1-\frac{1}{r})p \le (1-\frac{1}{s})q < 0$. From this, it follows that

$$\frac{(1-\frac{1}{s})q}{(1-\frac{1}{r})p} \ge 1$$

We shall also need the following lemmas in the proofs of our main results in this section.

Lemma 3.1 *Let $k(x,y) \ge 0$ be defined in $\triangle = \{(x,y) \in \Re^2 : y < x\}$ with k(x,y) nondecreasing in y and nonincreasing in x. Then the following inequality holds*

$$\begin{aligned} \textstyle\int_{-\infty}^{x} k(x,y)^{(1-\beta)(1-1/r)p'} v(y)^{-(1-1/r)p'} h(y)^{-(1-1/r)p'} dy \\ \ge \frac{(1-1/r)p' + (1-1/s)q}{(1-1/s)q} h(x)^{(1-1/s)q} \end{aligned} \qquad (3.20)$$

Proof: For any $x \in \Re$, let

$$\begin{aligned} J &= \int_{-\infty}^{x} k(x,y)^{(1-\beta)(1-1/r)p'} v(y)^{-(1-1/r)p'} h(y)^{-(1-1/r)p'} dy \\ &= \int_{-\infty}^{x} k(x,y)^{(1-\beta)(1-1/r)p'} v(y)^{-(1-1/r)p'} \\ &\quad \cdot \left(\int_{-\infty}^{y} k(y,z)^{(1-\beta)(1-1/r)p'} v(z)^{-(1-1/r)p'} dz \right)^{-\frac{(1-1/r)p'}{(1-1/r)p' + (1-1/s)q}} dy \quad \text{by } (1.3). \end{aligned}$$

Since $y < x$, it follows that $k(y,z) \ge k(x,z)$ and this is bounded above by

$$\begin{aligned} J &\ge \int_{-\infty}^{x} k(x,y)^{(1-\beta)(1-1/r)p'} v(y)^{-(1-1/r)p'} \\ &\quad \cdot \left(\int_{-\infty}^{y} k(x,z)^{(1-\beta)(1-1/r)p'} v(z)^{-(1-1/r)p'} dz \right)^{-\frac{(1-1/r)p'}{(1-1/r)p' + (1-1/s)q}} dy \\ &= \frac{(1-1/r)p' + (1-1/s)q}{(1-1/s)q} \left(\int_{-\infty}^{x} k(x,z)^{(1-\beta)(1-1/r)p'} v(z)^{-(1-1/r)p'} dz \right)^{\frac{(1-1/s)q}{(1-1/r)p' + (1-1/s)q}} \\ &\quad \text{by Lemma2.2} \\ &= \frac{(1-1/r)p' + (1-1/s)q}{(1-1/s)q} h(x)^{(1-1/s)q} \end{aligned}$$

This completes the proof of the Lemma.

Lemma 3.2 *Let $k(x,y) \geq 0$ be defined in $\triangle = \{(x,y) \in \Re^2 : y < x\}$ with k(x,y) nondecreasing in y and nonincreasing in x. Suppose (u(x),v(x)) satisfies the A(k,p,q,r,s) condition with constant C and if $(1-\frac{1}{r})^2 pp' - (1-\frac{1}{s})q^2 = 0$. Then*

$$\left(\int_y^\infty k(x,y)^{(1-1/s)\beta q} u(x)^{(1-1/s)q} h(x)^{(1-1/s)(1-1/r)p} dx\right)^{\frac{(1-1/s)q}{(1-1/r)p'}} \geq \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/r)p'}\right)^{\frac{(1-1/s)q}{(1-1/r)p'}}$$

$$\cdot C^{\frac{(1-1/s)^2(1-1/r)pq}{(1-1/r)p'+(1-1/s)q}} \left(\int_{-\infty}^y k(z,y)^{(1-1/s)\beta q} u(z)^{(1-1/s)q} dz\right)^{\frac{(1-1/s)q}{(1-1/r)p'+(1-1/s)q}} \qquad (3.21)$$

Proof: Let

$$\begin{aligned} J &= \int_y^\infty k(x,y)^{(1-1/s)\beta q} u(x)^{(1-1/s)q} h(x)^{(1-1/s)(1-1/r)p} dx \\ &= \int_y^\infty k(x,y)^{(1-1/s)\beta q} u(x)^{(1-1/s)q} \\ &\quad \cdot \left(\int_{-\infty}^x k(x,z)^{(1-\beta)(1-1/r)p'} v(z)^{-(1-1/r)p'} dz\right)^{\frac{(1-1/s)(1-1/r)p}{(1-1/r)p'+(1-1/s)q}} dy \ \ 1.3 \end{aligned} \qquad (3.22)$$

Since $(u(x), v(x))$ satisfies the A(k,p,q,r,s) condition with constant C, we have obtain from (2.4) that

$$\int_{-\infty}^x k(x,z)^{(1-\beta)(1-1/r)p'} v(z)^{-(1-1/r)p'} dz$$

$$\leq C^{(1-1/r)p'} \left(\int_x^\infty k(z,x)^{(1-1/s)\beta q} u(z)^{(1-1/s)q} dz\right)^{\frac{-(1-1/r)p'}{(1-1/s)q}} \qquad (3.23)$$

Substitute (3.23) into (3.22) and apply Lemma3.1 we have

$$\begin{aligned} J &\geq C^{\frac{(1-1/s)(1-1/r)pp'}{(1-1/r)p'+(1-1/s)q}} \int_y^\infty k(x,y)^{(1-1/s)\beta q} u(x)^{(1-1/s)q} \\ &\quad \cdot \left(\int_x^\infty k(z,x)^{(1-1/s)\beta q} u(z)^{(1-1/s)q} dz\right)^{\frac{-(1-1/s)q}{(1-1/r)p'+(1-1/s)q}} dx \\ &= \frac{(1-1/r)p'+(1-1/s)q}{(1-1/r)} C^{\frac{(1-1/s)(1-1/r)^2 pp'}{(1-1/r)p'+(1-1/s)q}} \\ &\quad \cdot \int_y^\infty k(x,y)^{(1-1/s)\beta q} u(x)^{(1-1/s)q} \left(\int_{-\infty}^y k(z,y)^{(1-1/s)\beta q} u(z)^{(1-1/s)q} dz\right)^{\frac{(1-1/r)p'}{(1-1/r)p'+(1-1/s)q}} \end{aligned}$$

Hence

$$J^{\frac{(1-1/s)q}{(1-1/r)p'}} \geq \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/r)p'}\right)^{\frac{(1-1/s)q}{(1-1/r)p'}} C^{\frac{(1-1/s)^2(1-1/r)pq}{(1-1/r)p'+(1-1/s)q}}$$
$$\cdot \left(\int_{-\infty}^{y} k(z,y)^{(1-1/s)\beta q} u(z)^{(1-1/s)q} dz\right)^{\frac{(1-1/s)q}{(1-1/r)p'+(1-1/s)q}}$$

and the proof is complete.

Theorem 3.1 *Let K be an integral operator defined in (1.1). Let $k(x,y) \geq 0$ is defined in $\triangle = \{(x,y) \in \Re^2 : y < x\}$ with k(x,y) nonincreasing in x and nondecreasing in y. If (u(x),v(x)) satisfies the A(k,p,q,r,s) condition with constant C and $(1-\frac{1}{r})^2 pp' - (1-\frac{1}{s})q^2 = 0$. Then*

$$\left(\int_{-\infty}^{\infty} \mid (Tf)(x)u(x) \mid^{(1-\frac{1}{s})q} dx\right)^{\frac{1}{(1-\frac{1}{s})q}} \geq AC^{(1-\frac{1}{s})} \left(\int_{-\infty}^{\infty} \mid f(x)v(x) \mid^{(1-\frac{1}{r})p} dx\right)^{\frac{1}{(1-\frac{1}{r})p}} \quad (3.24)$$

where

$$A = \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)q}\right)^{\frac{1}{(1-1/r)p'}} \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/r)p'}\right)^{\frac{1}{q}}$$

Proof: Assume $f \geq 0$ for which the right hand side of (3.25) is finite and let

$$\begin{aligned} J &= \int_{-\infty}^{\infty} \mid u(x)(Tf)(x) \mid^{(1-1/s)q} dx \\ &= \int_{-\infty}^{\infty} u(x)^{(1-1/s)q} \left(\int_{-\infty}^{x} k(x,y)f(y)dy\right)^{(1-1/s)q} dx \\ &= \int_{-\infty}^{\infty} u(x)^{(1-1/s)q} \left(\int_{-\infty}^{x} k(x,y)^{\beta} f(y)v(y)h(y)k(x,y)^{1-\beta} v(y)^{-1} h(y)^{-1} dy\right)^{(1-1/s)q} dx \end{aligned}$$

By Holder's inequality, we have

$J \geq \int_{-\infty}^{\infty} u(x)^{(1-1/s)q} \left(\int_{-\infty}^{x} k(x,y)^{(1-1/r)\beta p} [f(y)v(y)h(y)]^{(1-1/r)p} dy\right)^{\frac{(1-1/s)q}{(1-1/r)p}}$

$$\cdot \left(\int_{-\infty}^{x} k(x,y)^{(1-1/r)(1-\beta)p'} [v(y)h(y)]^{-(1-1/r)p'} dy\right)^{\frac{(1-1/s)q}{(1-1/r)p'}} dx$$

By Lemma3.1, we have

$$J \geq \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)q}\right)^{\frac{(1-1/s)q}{(1-1/r)p'}} \int_{-\infty}^{\infty} u(x)^{(1-1/s)q} \cdot \left(\int_{-\infty}^{x} k(x,y)^{(1-1/r)\beta p}[f(y)v(y)h(y)]^{(1-1/r)p}dy\right)^{\frac{(1-1/s)q}{(1-1/r)p}} h(x)^{(1-1/s)(1-1/r)p}$$

By Minkowskii's integral inequality (2.6), we obtain

$$J \geq \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)q}\right)^{\frac{(1-1/s)q}{(1-1/r)p'}} \left(\int_{-\infty}^{\infty}\left\{\int_{y}^{\infty} k(x,y)^{(1-1/s)\beta q}u(x)^{(1-1/s)q} \cdot h(x)^{(1-1/r)(1-1/s)p}dx\right\}^{\frac{(1-1/r)p}{(1-1/s)q}} [f(y)v(y)h(y)]^{(1-1/r)p}dy\right)^{\frac{(1-1/s)q}{(1-1/r)p}}$$

By Lemma3.2, we obtain

$$J \geq \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)q}\right)^{\frac{(1-1/s)q}{(1-1/r)p'}} \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)p'}\right)^{(1-1/s)} C^{\frac{(1-1/s)^2(1-1/r)^2pp'}{(1-1/r)p'+(1-1/s)q}} \cdot\left(\int_{\infty}^{\infty}[f(y)v(y)h(y)]^{(1-1/r)p}\left(\int_{y}^{\infty} k(y,z)^{(1-1/s)\beta q} \cdot u(z)^{(1-1/s)q}dz\right)^{\frac{(1-1/s)q}{(1-1/r)p'+(1-1/s)q}} dy\right)^{\frac{(1-1/s)q}{(1-1/r)p}} \tag{3.25}$$

Since (u(x),v(x)) satisfies the A(k,p,q,r,s) condition with constant C, we have from (2.4) that

$$\left(\int_y^\infty k(y,z)^{(1-1/s)\beta q}u(z)^{(1-1/s)q}dz\right)^{\frac{(1-1/s)q}{(1-1/r)p'+(1-1/s)q}} \geq C^{\frac{(1-1/s)(1-1/r)^2pp'}{(1-1/r)p'+(1-1/s)q}}\left(\int_{y}^{\infty} k(y,z)^{(1-1/r)(1-\beta)p'}v(z)^{-(1-1/r)p'}dz\right)^{-\frac{(1-1/r)p}{(1-1/r)p'+(1-1/s)q}} \tag{3.26}$$

Substitute (3.26) into (3.25) we obtain

$$J \geq \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)q}\right)^{\frac{(1-1/s)q}{(1-1/r)p'}} \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)p'}\right)^{(1-1/s)}$$

$$\cdot C^{\frac{(1-1/s)^2[(1-1/r)^2pp'+(1-1/r)p'q]}{(1-1/r)p'+(1-1/s)q}}\left(\int_{\infty}^{\infty}[f(y)v(y)h(y)]^{(1-1/r)p}\right.$$

$$\left.\cdot\left(\int_y^{\infty}k(y,z)^{(1-1/r)(1-\beta)p'}v(z)^{-(1-1/r)p'}dz\right)^{-\frac{(1-1/r)p}{(1-1/r)p'+(1-1/s)q}}dy\right)^{\frac{(1-1/s)q}{(1-1/r)p}}$$

By (1.3) we have

$$\begin{aligned} J &\geq \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)q}\right)^{\frac{(1-1/s)q}{(1-1/r)p'}}\left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)p'}\right)^{(1-1/s)} \\ &\cdot C^{\frac{(1-1/s)^2[(1-1/s)q^2+(1-1/r)p'q]}{(1-1/r)p'+(1-1/s)q}}\left(\int_{\infty}^{\infty}[f(y)v(y)h(y)]^{(1-1/r)p}h(y)^{(1-1/r)p}h(y)^{-(1-1/r)p}dy\right)^{\frac{(1-1/s)q}{(1-1/r)p}} \end{aligned}$$

Hence

$$\begin{aligned} J^{\frac{1}{(1-1/s)q}} &\geq \left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)q}\right)^{\frac{1}{(1-1/r)p'}}\left(\frac{(1-1/r)p'+(1-1/s)q}{(1-1/s)p'}\right)^{\frac{1}{q}} \\ &\cdot C^{(1-1/s)}\left(\int_{\infty}^{\infty}[f(y)v(y)]^{(1-1/r)p}dy\right)^{(1-1/r)p} \end{aligned}$$

This completes the proof of the theorem.

Remark 3.1 *In the limit $r\to\infty$ and $s\to\infty$ in in Theorem 3.1, we obtain*

$$\left(\int_{-\infty}^{\infty}|\,(Kf)(x)u(x)\,|^q\,dx\right)^{\frac{1}{q}} \geq AC\left(\int_{-\infty}^{\infty}|\,f(x)v(x)\,|^p\,dx\right)^{\frac{1}{p}} \tag{3.27}$$

where

$$A = \left(\frac{p'+q}{q}\right)^{\frac{1}{p'}}\left(\frac{p'+q}{p'}\right)^{\frac{1}{q}}$$

which is a new result.

Corollary 3.1 *In Theorem 3.1, if we set $q^2=4(1-\frac{1}{s})$ and $p=p'=2$, we obtain $r=s\to\infty$ and so (3.25) reduces to*

$$\left(\int_{-\infty}^{\infty}|\,(Tf)(x)u(x)\,|^2\,dx\right)^{\frac{1}{2}} \geq C\left(\int_{-\infty}^{\infty}|\,f(x)v(x)\,|^2\,dx\right)^{\frac{1}{2}} \tag{3.28}$$

Theorem 3.2 *Let K^* be an integral operator defined in (1). Let $k(x,y) \geq 0$ is defined in $\triangle = \{(x,y) \in \Re^2 : y < x\}$ with k(x,y) nonincreasing in x and nondecreasing in y. If (u(x),v(x)) satisfies the $A^*(k,p,q,r,s)$ condition with constant C^* and $(1-\frac{1}{r})^2 pp' - (1-\frac{1}{s})q^2 = 0$. Then*

$$\left(\int_{-\infty}^{\infty} \mid (K^*f)(x)u(x) \mid^{(1-\frac{1}{s})q} dx\right)^{\frac{1}{(1-\frac{1}{s})q}} \geq AC^{*(1-\frac{1}{s})} \left(\int_{-\infty}^{\infty} \mid f(x)v(x) \mid^{(1-\frac{1}{r})p} dx\right)^{\frac{1}{(1-\frac{1}{r})p}} \quad (3.29)$$

where

$$A = \left(\frac{(1-1/r)p' + (1-1/s)q}{(1-1/s)q}\right)^{\frac{1}{(1-1/r)p'}} \left(\frac{(1-1/r)p' + (1-1/s)q}{(1-1/r)p'}\right)^{\frac{1}{q}}$$

Proof: The proof is similar to that of Theorem 3.1 except that we define h(y) by

$$h(y) = \left(\int_{y}^{\infty} K(y,z)^{(1-\beta)(1-1/r)p'} v(z)^{-(1-1/r)p'} dz\right)^{\frac{-1}{(1-1/r)p+(1-1/s)q}}$$

Remark 3.2 *If in Theorem 3.2 we let $r \to \infty$ and $s \to \infty$,then our result reduces to*

$$\left(\int_{-\infty}^{\infty} \mid (K^*f)(x)u(x) \mid^{q} dx\right)^{\frac{1}{q}} \geq AC^{*} \left(\int_{-\infty}^{\infty} \mid f(x)v(x) \mid^{p} dx\right)^{\frac{1}{p}} \quad (3.30)$$

where

$$A = \left(\frac{p'+q}{q}\right)^{\frac{1}{p'}} \left(\frac{p'+q}{p'}\right)^{\frac{1}{q}}$$

References

[1] K. F. Andersen and H. P.Heinig

Weighted norm inequalities for certain integral operators. Siam. J. Math. Anal., 14(4),(1983),834–844.

[2] P. R. Beasack and H. P. Heining

Hardy's inequalities with indices less than 1 Proc. Amer Math. Soc.83(1981),532–536.

[3] J. S. Bradley

Hardy's inequalities with mixed norms. Canad. Math. Bull. 21(1978),405–408.

[4] G. H. Hardy, J. E. Littlewood and G. Polya

Inequalities Cambridge University Press, 1959.

[5] J. A. Oguntuase and R. Adegoke

Weighted norm inequalities for some integral operators. Coll. of Sci. Pap. Fac. Sci. Univ. Kragejuvac 21(1999), 55–62.

[6] J. A. Oguntuase and C. O. Imoru

New generalizations of Hardy's integral inequality. J. Math. Anal. Appl. 240 (1999).