ON NEW PROOFS OF INEQUALITIES INVOLVING TRIGONOMETRIC FUNCTIONS

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ABSTRACT. In the note, some new proofs for inequalities invoving trigonometric functions are given.

1. Introduction

In [9], J. B. Wilker proposed that

(a) If $0 < x < \frac{\pi}{2}$, then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{1}$$

(b) There exists a largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x \tag{2}$$

for $0 < x < \frac{\pi}{2}$.

In [8], the inequality (1) was proved, and the following inequalities were also obtained

$$2 + \frac{8}{45}x^3 \tan x > \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x. \tag{3}$$

The constants $\frac{8}{45}$ and $\left(\frac{2}{\pi}\right)^4$ are best possible, that is, they can not be replaced by smaller or larger numbers respectively.

The inequalities in (1) and (3) are called Wilker's inequalities in [3].

In this note, we will give new proofs for the inequalities in (1) and (3).

2. A New Proof of Inequality (1)

The inequality (1) can be rewritten as

$$\sin^2 x \cos x + x \sin x > 2x^2 \cos x. \tag{4}$$

Let

$$g(x) = \sin^2 x \cos x + x \sin x - 2x^2 \cos x, \quad x \in \left(0, \frac{\pi}{2}\right), \tag{5}$$

$$h(x) = 2 \sin x \cos^2 x - 3x \cos x + (1 + x^2) \sin x, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{6}$$

Direct calculation yields

$$g'(x) = 2 \sin x \cos^2 x - \sin^3 x + \sin x + x \cos x - 4x \cos x + 2x^2 \sin x$$

Date: April 18, 2000.

2000 *Mathematics Subject Classification.* Primary 26D05.

Key words and phrases. Trigonometric function, Wilker's inequality, Bernoulli number.

The first and the fifth authors were supported in part by NSF of Henan Province (no. 004051800), SF for Pure Research of the Education Committee of Henan Province (no. 1999110004), and Doctor Fund of Jiaozuo Institute of Technology, The People's Republic of China.

$$
\begin{aligned}
&= (x^2 - \sin^2 x)\sin x + 2\sin x\cos^2 x - 3x\cos x + (1+x^2)\sin x \\
&= (x^2 - \sin^2 x)\sin x + h(x), \\
h'(x) &= 2\cos^3 x - 4\sin^2 x\cos x - 3\cos x + 3x\sin x + 2x\sin x + (1+x^2)\cos x \\
&= (x^2 - \sin^2 x)\cos x + 5(x - \sin x\cos x)\sin x.
\end{aligned}
$$

Since $x > \sin x$ for $x > 0$, we have $h'(x) > 0$, $h(x)$ is increasing. From $h(0) = 0$, we obtain $h(x) > 0$, and then $g'(x) = (x^2 - \sin^2 x)\sin x + h(x) > 0$, the function $g(x)$ is increasing. From $g(0) = 0$ and $g\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$, we get $0 < g(x) < \frac{\pi}{2}$ for $x \in \left(0, \frac{\pi}{2}\right)$.

The proof of inequality (1) is complete.

3. A New Proof of Inequalities in (3)

Define

$$
\psi(x) = \frac{\sin 2x}{2x^5} + \frac{1}{x^4} - \frac{2\cot x}{x^3} \tag{7}
$$

for $0 < x < \frac{\pi}{2}$. Easy computation yields

$$
\psi'(x) = -\frac{5\sin 2x}{2x^6} + \frac{\cos 2x}{x^5} - \frac{4}{x^5} + \frac{6\cos x}{x^4\sin x} + \frac{2}{x^3\sin^2 x}. \tag{8}
$$

It is well-known [1, p. 226–227] that

$$
\sin 2x = 2x - \frac{4}{3}x^3 + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+5}}{(2n+5)!} x^{2n+5}, \tag{9}
$$

$$
\cot x = \frac{1}{x} - \frac{1}{3}x - \sum_{n=0}^{\infty} \frac{2^{2n+4} B_{n+2}}{(2n+4)!} x^{2n+3}, \tag{10}
$$

where B_n denotes the n-th Bernoulli number, which is defined in [1, p. 228] by

$$
\frac{t}{\mathrm{e}^t - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} B_k}{(2k)!} t^{2k}, \quad |t| < 2\pi. \tag{11}
$$

Therefore, by direct computation, we have

$$
\psi(x) = \sum_{n=0}^{\infty} \frac{2^{2n+4}}{(2n+5)!} \left\{2(2n+5)B_{n+2} + (-1)^n\right\} x^{2n}. \tag{12}
$$

From the identity in [1, p. 231]

$$
\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{\pi^{2n} \cdot 2^{2n-1}}{(2n)!} B_n, \tag{13}
$$

by mathematical induction, for $n > 2$, we have

$$
2(2n+5)B_{n+2} = \frac{4\cdot(2n+5)!}{(2\pi)^{2n+4}} \sum_{k=1}^{\infty} \frac{1}{k^{2n+4}} > \frac{4\cdot(2n+5)!}{(2\pi)^{2n+4}} > 1, \tag{14}
$$

then $\phi''(x) \geqslant 0$, where $\phi(x) = \psi(\sqrt{x})$, and $\phi'(x)$ is increasing on $(0, \frac{\pi^2}{4})$. Since $\phi'\left(\left(\frac{\pi}{2}\right)^2\right) = \psi'\left(\frac{\pi}{2}\right) = 2\cdot\left(\frac{2}{\pi}\right)^3 \cdot \left(1 - \frac{10}{\pi^2}\right) < 0$, hence $\phi'(x) < 0$, and then $\phi(x)$ is decreasing, that is $\psi(x)$ is decreasing on $(0, \frac{\pi}{2})$, then we have

$$
\frac{8}{45} = \psi(0) > \psi(x) > \psi\left(\frac{\pi}{2}\right) = \frac{16}{\pi^4}, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{15}
$$

Inequalities in (15) are equivalent to those in (3). The proof of inequalities in (3) is complete.

Remark 1. For details about Bernoulli numbers, also please refer to [2, 5, 7].

Remark 2. Using Tchebysheff's integral inequality, many inequalities involving the function $\frac{\sin x}{x}$ are constructed in [6].

References

[1] Group of Compilation, *Handbook of Mathematics*, People's Education Press, Beijing, China, 1979. (Chinese)

[2] Sen-Lin Guo and Feng Qi, *Recursion formulae for* $\sum_{m=1}^{n} m^k$, Zeitschrift für Analysis und ihre Anwendungen **18** (1999), no. 4, 1123–1130.

[3] J.-C. Kuang, *Applied Inequalities* (*Changyong Budengshi*), 2nd edition, Hunan Education Press, Changsha, China, 1993, page 349. (Chinese)

[4] Heng-Qiang Lin, Jian-Wu Wang, and Pan-Zhou Wang, *Proof for one of inequality*, Communicated paper on The 7-th Conference of Teaching and Research for Universities and Colleges of Henan Province, July 1999, unpublished.

[5] Feng Qi, *Generalized Bernoulli polynomial*, submitted to Mathematics and Informatics Quarterly.

[6] Feng Qi, Li-Hong Cui, and Sen-Lin Xu, *Some inequalities constructed by Tchebysheff's integral inequality*, Mathematical Inequalities and Applications **2** (1999), no. 4, 517–528.

[7] Yong-Huan Shen, Zai-Zhong Liang, Lü-Hu Xu, and Qian-Qian Cai, *Practical Handbook of Mathematics* (*Shíyòng Shùxué Shŏucè*), Science Press, Beijing, China, 1992; Second Print, 1997; pp. 273–274, 778–780. (Chinese)

[8] J. S. Sumner, A. A. Jagers, M. Vowe, and J. Anglesio, *Inequalities involving trigonometric functions*, The American Mathematical Monthly **98** (1991), no. 3, 264–267.

[9] J. B. Wilker, *E 3306*, The American Mathematical Monthly **96** (1989), no. 1, 55.

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