GAUSS–PÓLYA TYPE RESULTS AND THE HÖLDER INEQUALITY

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Abstract. Some special Gauss–Pólya type inequalities are obtained by the use of Hölder's inequality.

1. Introduction

Hölder's inequality is a basic tool in analysis. In its discrete form, it states the following.

Theorem 1. *Let $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ be positive n–tuples and p, q nonzero real numbers satisfying $p^{-1} + q^{-1} = 1$. If $p, q > 0$ we have*

$$\sum_{k=1}^{n} a_k b_k \le \left(\sum_{k=1}^{n} a_k^p \right)^{1/p} \left(\sum_{k=1}^{n} b_k^q \right)^{1/q}.$$

If either p or q is negative, the inequality is reversed.

The integral form is similar.

Theorem 2. *Suppose f and g are real functions defined on an interval $[a, b]$ and such that $|f|^p$ and $|q|^q$ are integrable on $[a, b]$. If $p > 1$ and $p^{-1} + q^{-1} = 1$, then*

$$\int_a^b |f(x)g(x)|dx \le \left(\int_a^b |f(x)|^p dx \right)^{1/p} \left(\int_a^b |g(x)|^q dx \right)^{1/q}.$$

A number of extensions and generalizations of these results are given in [3]).

In their work "Problems and Theorems in Analysis" [6], Pólya and Szegö gave two cognate theorems which were to become seminal. The first is similar to Hölder's inequality.

Theorem 3. *Suppose $f : [0, \infty) \to \mathbb{R}$ is a nonnegative and nonincreasing function and u, v nonnegative real numbers. Then provided the integrals exist,*

$$\left(\int_0^\infty x^{u+v} f(x)dx \right)^2 \le \left(1 - \left(\frac{u-v}{u+v+1} \right)^2 \right) \int_0^\infty x^{2u} f(x)dx \int_0^\infty x^{2v} f(x)dx.$$

The other, remarkably, proceeds in the opposite direction.

1991 *Mathematics Subject Classification.* Primary 26D10, Secondary 60E15.

Key words and phrases. Gauss–Pólya inequalities, Hölder inequality.

Theorem 4. *Suppose $f : [0,1] \to \mathbb{R}$ is a nonnegative and nondecreasing function and u, v nonnegative real numbers. Then*

$$\left(\int_0^1 x^{u+v} f(x)dx \right)^2 \geq \left(1 - \left(\frac{u-v}{u+v+1} \right)^2 \right) \int_0^1 x^{2u} f(x)dx \int_0^1 x^{2v} f(x)dx.$$

These and a number of related results have applications in probability, since by scaling so that $\int f(x)dx = 1$, we can interpret f as a probability density. To give their flavour, suppose X is a random variable with nondecreasing density f on $[0,1]$. Theorem 4 provides

$$\left[E\left(X^{u+v} \right) \right]^2 \geq \left[1 - \left(\frac{u-v}{u+v+1} \right)^2 \right] E(X^{2u}) E(X^{2v}).$$

Thus when $u = 1$ and $v = 0$, we have $[E(X)]^2 \geq (3/4)E(X^2)$. There is, of course, also the standard inequality $E(X^2) \geq [E(X)]^2$ for any random variable possessing a second moment. Thus for a random variable on $[0,1]$ with a nondecreasing density function, we have the two–sided inequality

$$E(X^2) \geq [E(X)]^2 \geq (3/4)E(X^2).$$

For a comprehensive overview of results relating to Theorems 3 and 4 up to 1984 see Beesack [2].

In 1990, Alzer [1] discovered a surprising and elegant pathway to further generalizations, the so–called Gauss–Pólya inequalities. The ideas behind this breakthrough are suggested by the reformulation of the result of Theorem 4 in terms of derivatives as

$$\left[\int_0^1 \left(\frac{d}{dx} x^{u+v+1} \right) f(x)dx \right]^2 \geq \int_0^1 \left(\frac{d}{dx} x^{2u+1} \right) f(x)dx \cdot \int_0^1 \left(\frac{d}{dx} x^{2v+1} \right) f(x)dx.$$

This has stimulated a variety of research, see, for example, Varošanec, Pečarić and Šunde [7] and Pearce, Pečarić and Šunde [5], who have found a number of extensions. The generality of these ideas is indicated by the fact that there exist also operator versions of at least some of them, as found in further work with Mond [4].

In this note we develop a general but simple theorem of this type based on the discrete Hölder inequality. In Section 2 we derive our main result, which involves sums, integrals, derivatives and a number of free functions and parameters. In Section 3 we deduce some interesting special cases, first by making particular choices of parameters and then by making particular choices of functions.

2. Results

Our general result is as follows.

Theorem 5. *Let $[a,b]$ be a finite interval, $f : [a,b] \to \mathbb{R}$ a nonnegative and monotone function and $x_i : [a,b] \to \mathbb{R}$ $(i = 1, ..., n)$ functions with continuous first derivatives. Suppose that $p, q > 1$ with $p^{-1} + q^{-1} = 1$ and p_i $(i = 1, ..., n)$ are positive real numbers satisfying $\sum_{i=1}^n p_i = 1$. For $t \in [a,b]$, set*

$$F(t) := f(t) \left[\left(\sum_{i=1}^n |x_i(t)|^q \right)^{1/q} - \left(\sum_{i=1}^n p_i^p \right)^{-1/p} \sum_{j=1}^n p_j x_j(t) \right].$$

Then

$$\left(\sum_{i=1}^{n}\left|\int_a^b x_i'(t)f(t)dt\right|^q\right)^{1/q} \geq \pm\left[\int_a^b \left(\left[\sum_{i=1}^{n}|x_i(t)|^q\right]^{1/q}\right)' f(t)dt + F(a) - F(b)\right],$$

the plus sign applying when f is nondecreasing and the minus when f is nonincreasing.

Proof. Since $x_i \in C^1[a,b]$ $(i = 1,\dots,n)$ and f is monotone, the integrals in the enunciation exist. From integration by parts we have

$$\begin{aligned}\sum_{i=1}^{n} p_i \int_a^b {x_i}'(t)f(t)dt &= f(b)\sum_{i=1}^{n} p_i x_i(b) - f(a)\sum_{i=1}^{n} p_i x_i(a)\\ &\quad - \int_a^b \left(\sum_{i=1}^{n} p_i x_i(t)\right) df(t). \qquad (2.1)\end{aligned}$$

Also, Hölder's discrete inequality provides

$$\begin{aligned}\pm\sum_{i=1}^{n} p_i \int_a^b x_i'(t)f(t)dt &\leq \sum_{i=1}^{n} p_i \int_a^b |x_i'(t)f(t)|dt\\ &\leq \left(\sum_{i=1}^{n} p_i^p\right)^{1/p} \left(\sum_{j=1}^{n}\left|\int_a^b x_j'(t)f(t)dt\right|^q\right)^{1/q} \qquad (2.2)\end{aligned}$$

and

$$\sum_{i=1}^{n} p_i x_i(t) \leq \sum_{i=1}^{n} |p_i x_i(t)| \leq \left(\sum_{i=1}^{n} p_i^p\right)^{1/p} \left(\sum_{j=1}^{n} |x_j(t)|^q\right)^{1/q} \quad \text{for all} \quad t \in [a,b].$$

The latter inequality yields

$$\int_a^b \left(\sum_{i=1}^{n} p_i x_i(t)\right) df(t) \leq \left(\sum_{i=1}^{n} p_i^p\right)^{1/p} \int_a^b \left(\sum_{j=1}^{n} |x_j(t)|^q\right)^{1/q} df(t)$$

when f is nondecreasing and the reverse inequality when f is nonincreasing.

From this result, (2.1) and (2.2) we derive

$$\begin{aligned}&\left(\sum_{i=1}^{n}\left|\int_a^b x_i'(t)f(t)dt\right|^q\right)^{1/q}\\ &\quad\geq \pm\left[\left(\sum_{i=1}^{p} p_i^p\right)^{-1/p}\left\{f(b)\sum_{j=1}^{n} p_j x_j(b) - f(a)\sum_{j=1}^{n} p_j x_j(a)\right\}\right.\\ &\qquad\left. - \int_a^b \left(\sum_{i=1}^{n} |x_i(t)|^q\right)^{1/q} df(t)\right].\end{aligned}$$

A further integration by parts gives that

$$\int_a^b \left(\sum_{j=1}^n |x_j(t)|^q\right)^{1/q} df(t) = \left(\sum_{j=1}^n |x_j(b)|^q\right)^{1/q} f(b) - \left(\sum_{j=1}^n |x_j(a)|^q\right)^{1/q} f(a) - \int_a^b \left[\left(\sum_{j=1}^n |x_j(t)|^q\right)^{1/q}\right]' f(t)dt.$$

Combining this and the previous displayed relation and rearranging establishes the desired result. □

3. Special cases

For brevity we note only results applying with the case that f is nondecreasing. First we address particular choices of parameter.

The choice $p = q = 2$ gives

$$\begin{aligned} &\left(\sum_{i=1}^n \left|\int_a^b x_i'(t) f(t)dt\right|^2\right)^{1/2} \\ &\quad + f(b)\left[\left(\sum_{i=1}^n |x_i(b)|^2\right)^{1/2} - \left(\sum_{i=1}^n p_i^2\right)^{-1/2} \sum_{j=1}^n p_j x_j(b)\right] \\ &\geq \int_a^b \left(\sqrt{\sum_{i=1}^n |x_i(t)|^2}\right)' f(t)dt \\ &\quad + f(a)\left[\left(\sum_{i=1}^n |x_i(a)|^2\right)^{1/2} - \left(\sum_{i=1}^n p_i^2\right)^{-1/2} \sum_{j=1}^n p_j x_j(a)\right], \end{aligned}$$

which is related to the Cauchy–Schwarz result.

Another natural choice is $p_i = 1/n$ $(1 \leq i \leq n)$, for which

$$\begin{aligned} &\left(\sum_{i=1}^n \left|\int_a^b x_i'(t) f(t)dt\right|\right)^{\frac{1}{q}} + f(b)\left[\left(\sum_{i=1}^n |x_i(b)|^q\right)^{\frac{1}{q}} - n^{-\frac{1}{p}} \sum_{i=1}^n x_i(b)\right] \\ &\geq \int_a^b \left(\left[\sum_{i=1}^n |x_i(t)|^q\right]^{\frac{1}{q}}\right)' f(t)dt + f(a)\left[\left(\sum_{i=1}^n |x_i(a)|^q\right)^{\frac{1}{q}} - n^{-\frac{1}{p}} \sum_{i=1}^n x_i(a)\right]. \end{aligned}$$

In the context of recent work on Gauss–Pólya inequalities, a common assumption is $x_i(a) = A$, $x_i(b) = B$ $(i = 1, \dots, n)$. In the event that this holds, we derive

$$\left(\sum_{i=1}^{n} \left| \int_a^b x_i'(t) f(t) dt \right|^q \right)^{1/q} + f(b) \left[n^{\frac{1}{q}} |B| - \left(\sum_{i=1}^{n} p_i^p \right)^{-1/p} B \right]$$
$$\geq \int_a^b \left(\left[\sum_{i=1}^{n} |x_i(t)|^q \right]^{1/q} \right)' f(t) dt + f(a) \left[n^{\frac{1}{q}} |A| - \left(\sum_{i=1}^{n} p_i^p \right)^{-1/p} A \right].$$

If we further set $p_i = 1/n$ in this inequality, there is a simplication to

$$n^{-(p-1)/p} \left(\sum_{i=1}^{n} \left| \int_a^b x_i'(b) f(t) dt \right|^q \right)^{1/q} + f(b) \left[|B| - B \right]$$
$$\geq n^{-(p-1)/p} \int_a^b \left(\left(\sum_{i=1}^{n} |x_i(t)|^q \right)^{1/q} \right)' f(t) dt + f(a) \left[|A| - A \right].$$

Finally, if the common endpoint values A, B are both nonnegative, then

$$\left(\sum_{i=1}^{n} \left| \int_a^b x_i'(t) f(t) dt \right|^q \right)^{1/q} \geq \int_a^b \left(\left(\sum_{i=1}^{n} |x_i(t)|^q \right)^{1/q} \right)' f(t) dt,$$

which holds for all $q > 1$.

We now consider the particular choices $a = 0$, $b = 1$, $n = 2$, $x_1(t) = t^u$, $x_2(t) = t^v$, where $u, v > 0$. This yields

$$\left(u^q \left[\int_0^1 t^{u-1} f(t) dt \right]^q + v^q \left[\int_0^1 t^{v-1} f(t) dt \right]^q \right)^{1/q}$$
$$+ f(1) \left[2^{1/q} - (p_1^p + p_2^p)^{-1/p} \right]$$
$$\geq \int_0^1 \left(t^{uq} + t^{vq} \right)^{-1/p} \left(u t^{uq-1} + v t^{vq-1} \right) f(t) dt.$$

With the specific values $p_1 = p_2 = 1/2$, we get

$$\left[u^q \left(\int_0^1 t^{u-1} f(t) dt \right)^q + v^q \left(\int_0^1 t^{v-1} f(t) dt \right)^q \right]^{1/q} \geq \int_0^1 \frac{u t^{uq-1} + v t^{vq-1}}{\left(t^{uq} + t^{vq} \right)^{1/p}} f(t) dt.$$

Acknowledgement. The authors thank the referee for helpful comments.

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