A CONVOLUTED FIBONACCI SEQUENCE - PART I

A. SOFO

ABSTRACT. We consider a generalisation of the classical Fibonacci sequence, and by the use of function theoretic methods, generate binomial type series which may be expressed in closed form. Some new identities are also given.

1. Introduction

In this paper we will consider a generalised Fibonacci sequence, which we shall call a convoluted Fibonacci sequence, and generate many finite binomial type sums which, by the use of residue theory may be expressed in closed form. To quote from the experience of Hilton and Pederson [3]: '*Students respond enthusiastically to the stimuli provided by the geometric-algebraic patterns which they may find in the Pascal Triangle of binomial coefficients and by the various relations among the classical Fibonacci numbers'.* We shall therefore extend some results of the classical Fibonacci polynomials. In Section 2 we consider an arbitrary order difference scheme and by the use of Z transform theory generate binomial type sums that may be represented in closed form. In Section 3 we identify some well known polynomials which may also be expressed in terms of finite products, and recover some results quoted by Binz [1].

2. Technique

Consider what we shall describe as a generalised, or convoluted, Fibonacci sequence f_n, that satisfies

$$\left.\begin{array}{r}\sum_{j=0}^{R}\binom{R}{R-j}(-c)^{R-j}\sum_{r=0}^{j}\binom{j}{r}(-b)^{j-r}f_{n+r-(R-j)a}=w_n;\ n\geq aR\\ \sum_{r=0}^{R}\binom{R}{r}(-b)^{R-r}f_{n+r}=w_n;\ \ n<aR\end{array}\right\} \tag{2.1}$$

with a and R integer, b and c real and w_n is a discrete forcing term. A method of analyzing the solution of system (2.1) is by the use of Z transform techniques. Without loss of generality let $w_n=0,\ f_{R-1}=1$ and all other initial conditions of the system (2.1) be zero. If we now take the Z transform of (2.1), utilize the two Z transform properties

$$Z\left[f_{n+k}\right]=z^k\left[f\left(z\right)-\sum_{n=0}^{k-1}f_nz^{-n}\right]$$

and

$$Z\left[f_{n-k}U_{n-k}\right]=z^{-k}F\left(z\right),$$

Date: April, 2000.

1991 *Mathematics Subject Classification.* Primary 05A10, 33C20, 40A25.

where U_{n-k} is the discrete step function, we obtain

$$F(z)\left\{\sum_{j=0}^{R}\binom{R}{j}(z-b)^{j}\left(-cz^{-a}\right)^{R-j}\right\}=z. \tag{2.2}$$

From (2.2)

$$F(z)=\frac{z}{\left(z-b-cz^{-a}\right)^{R}}=\frac{z^{aR+1}}{\left(z^{a+1}-bz^{a}-c\right)^{R}}. \tag{2.3}$$

In series form, (2.3) may be expressed as

$$F(z)=\sum_{r=0}^{\infty}\binom{R+r-1}{r}\frac{c^{r}z^{1-ar}}{(z-b)^{R+r}} \tag{2.4}$$

and we may obtain the inverse Z transform of (2.4) such that

(2.5)

$$f_{n}(a,b,c,R)=f_{n}=\sum_{r=0}^{\left[\frac{n+1-R}{a+1}\right]}\binom{R+r-1}{r}\binom{n-ar}{R+r-1}\left(\frac{c}{b}\right)^{r}b^{n-ar-R+1}$$

where $[x]$ represents the integer part of x. The inverse Z transform of (2.3) may also be expressed as

$$f_{n}=\frac{1}{2\pi i}\oint_{C}z^{n}\left(\frac{F(z)}{z}\right)dz=\sum_{j=0}^{a}z^{n}\mathrm{Res}_{j}\left(\frac{F(z)}{z}\right), \tag{2.6}$$

where C is a smooth Jordan curve enclosing the singularities of (2.3) and Res_{j} is the residue of the poles of (2.3). The residue, Res_{j}, of (2.6) depend on the zeros of the characteristic function in (2.3), namely

$$g(z)=z^{a+1}-bz^{a}-c. \tag{2.7}$$

Now, $g(z)$ has $a+1$ distinct zeros ξ_{j}, $j=0,1,2,3,...,a$, for

$$c\neq-a^{a}\left(\frac{b}{a+1}\right)^{a+1}$$

therefore the singularities in (2.3) are all poles of order R. We may now write (2.6) as

$$f_{n}=\sum_{j=0}^{a}\sum_{\mu=0}^{R-1}Q_{R,\mu}\left(\xi_{j}\right)\binom{n}{R-1-\mu}\xi_{j}^{n-R+1+\mu} \tag{2.8}$$

where

$$\mu!Q_{R,\mu}\left(\xi_{j}\right)=\lim_{z\to\xi_{j}}\left[\frac{d^{\mu}}{dz^{\mu}}\left\{\left(z-\xi_{j}\right)^{R}\frac{F(z)}{z}\right\}\right] \tag{2.9}$$

for each $j=0,1,2,3,...,a$, and $F(z)$ is given by (2.3). Combining the expressions in (2.5) and (2.8) we have that

$$\sum_{r=0}^{\left[\frac{n+1-R}{a+1}\right]}\binom{R+r-1}{r}\binom{n-ar}{R+r-1}\left(\frac{c}{b}\right)^{r}b^{n-ar-R+1}$$

$$= \sum_{j=0}^{a} \sum_{\mu=0}^{R-1} Q_{R,\mu}\left(\xi_j\right) \binom{n}{R-1-\mu} \xi_j^{n-R+1+\mu} \tag{2.10}$$

and putting $n = n^* (a+1) + R - 1$ in (2.10) and renaming n^* as n, we have an alternate form

$$\sum_{r=0}^{n} \binom{R+r-1}{r} \binom{n(a+1)+R-1-ar}{R+r-1} \left(\frac{c}{b}\right)^r b^{n(a+1)-ar}$$

$$= \sum_{j=0}^{a} \sum_{\mu=0}^{R-1} Q_{R,\mu}\left(\xi_j\right) \binom{n(a+1)+R-1}{R-1-\mu} \xi_j^{n(a+1)+\mu}. \tag{2.11}$$

3. Fibonacci related polynomials and products

Consider (2.1) for $a = 1$, and $R = 1$ such that

$$f_{n+1} - b f_n - c f_{n-1} = 0,\ f_0 = 1 \tag{3.1}$$

and a solution to (3.1) may be written as

$$f_n = \sum_{r=0}^{[n/2]} \binom{n-r}{r} c^r b^{(n-2r)}. \tag{3.2}$$

Some polynomials of f_n are given on page 49 of the excellent article by Hilton and Pedersen [3]. The recurrence (3.1), for some parameter values b and c may be identified as shown in the following table.

b	c	name	generating function	zeros	solution of (3.1)
1	1	Fibonacci	$z^2 - z - 1$	$\frac{1 \pm \sqrt{5}}{2}$	$\frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right\}$
1	2	Jacobsthal	$z^2 - z - 2$	$2, -1$	$\frac{1}{3}\left(2^n - (-1)^n\right)$
2	1	Pell	$z^2 - 2z - 1$	$1 \pm \sqrt{2}$	$\frac{1}{\sqrt{2}}\left\{\left(1+\sqrt{2}\right)^{n+1} - \left(1-\sqrt{2}\right)^{n+1}\right\}$
3	-2	Fermat	$z^2 - 3z + 2$	$2, 1$	$2^{n+1} - 1$
2	-1	Chebyshev	$z^2 - 2z + 1$	$1, 1$	$n + 1$

Horadam [4] and [5] has recently written a lucid and pertinent examination of Jacobsthal representation numbers and polynomials, and functional forms.

Hendel and Cooke [2] suggest that second order recursions may be represented by finite products involving trigonometric functions. From (3.2) we may thus write

$$f_n = \prod_{j=1}^{n} \left\{ b - 2i\sqrt{c}\cos\left(\frac{\pi j}{n+1}\right) \right\}$$

and so after multiplying corresponding factors

$$f_n = b^{n-2[n/2]} \prod_{j=1}^{[n/2]} \left\{ b^2 + 4c\cos^2\left(\frac{\pi j}{n+1}\right) \right\}.$$

We can see that for $b=0$ and n even

$$f_n = (4c)^n \prod_{j=1}^{n} \cos^2\left(\frac{\pi j}{2n+1}\right)$$

and since we have that $f_n = c^n$ we may deduce

$$\prod_{j=1}^{n} \cos\left(\frac{\pi j}{2n+1}\right) = 2^{-n}.$$

The characteristic function (2.7), for $a=1$, has two (distinct) zeros, $2\xi_{0,1} = b \pm \sqrt{b^2+4c}$, from (3.2) and (2.6)

$$\sum_{r=0}^{[n/2]} \binom{n-r}{r} c^r b^{(n-2r)} = \frac{1}{\sqrt{b^2+4c}} \left\{ \left(\frac{b+\sqrt{b^2+4c}}{2}\right)^{n+1} - \left(\frac{b-\sqrt{b^2+4c}}{2}\right)^{n+1} \right\} \tag{3.3}$$

and putting $b=(x-1)^2$ and $c=bx$, we obtain the result obtained by Binz [1]; namely

$$\sum_{r=0}^{[n/2]} \binom{n-r}{r} x^r (x-1)^{2n-2r} = \frac{1}{x^2-1} \left\{ (x(x-1))^{n+1} - (1-x)^{n+1} \right\}.$$

We can integrate, for $0 \le x \le 1$, the result given by Binz , therefore producing the new identity

$$\sum_{r=0}^{[n/2]} \binom{n-r}{r} B(r+1, 2n-2r+1) = \frac{{}_2F_1\left[\begin{matrix}1,1\\ n+2\end{matrix}\middle| -1\right]}{n+1} + \frac{(-1)^n\, n!\sqrt{\pi}\; {}_2F_1\left[\begin{matrix}1,n+2\\ 2n+3\end{matrix}\middle| -1\right]}{2^{2n+2}\Gamma\left(n+\frac{3}{2}\right)},$$

where $B(x,y)$ is the Beta function, $\Gamma(x)$ is the Gamma function and ${}_2F_1[-]$ is the classical Gauss hypergeometric function. Differentiating (3.3) with respect to c and substituting we obtain, in an easier manner, yet another result quoted by Binz [1]

$$\sum_{r=0}^{[n/2]} r \binom{n-r}{r} x^r (x-1)^{2n-2r} = \frac{x(x-1)^n}{(x+1)^3} \left\{ \begin{matrix} (n+1)\, x \left\{x^{n-1} + (-1)^n\right\} + \\ (n-1)\left\{x^{n+1} + (-1)^n\right\} \end{matrix} \right\}.$$

We may derive (3.3) from a slightly different viewpoint and also in the process give a solution to a problem posed by Krafft and Schaefer [7]. Consider the two binomial expansions

$$\left(\frac{1+\sqrt{1+x}}{2}\right)^{2\mu} = 4^{-\mu} \sum_{r=0}^{2\mu} \binom{2\mu}{r} (1+x)^{r/2}; \tag{3.4}$$

and

$$\left(\frac{1-\sqrt{1+x}}{2}\right)^{2\mu} = 4^{-\mu}\sum_{r=0}^{2\mu}(-1)^r \binom{2\mu}{r}(1+x)^{r/2}. \tag{3.5}$$

Subtracting (3.5) from (3.4) we have

$$2^{-(2\mu-1)}\sum_{j=0}^{\mu}\binom{2\mu}{2j+1}(1+x)^j = \frac{\left(\frac{1+\sqrt{1+x}}{2}\right)^{2\mu}-\left(\frac{1-\sqrt{1+x}}{2}\right)^{2\mu}}{\sqrt{1+x}} \tag{3.6}$$

and to identify (3.6) with (3.3) let $2\mu = n+1$ so that

$$2^{-n}\sum_{j=0}^{\lceil\frac{n-1}{2}\rceil}\binom{n+1}{2j+1}(1+x)^j = \frac{\left(\frac{1+\sqrt{1+x}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{1+x}}{2}\right)^{n+1}}{\sqrt{1+x}} \tag{3.7}$$

where $\lceil\nu\rceil$ is the least integer not smaller than ν. From (3.3) let $b=1$ and $c=x/4$, hence

$$\begin{aligned}\sum_{r=0}^{[n/2]}\binom{n-r}{r}\left(\frac{x}{4}\right)^r &= \frac{\left(\frac{1+\sqrt{1+x}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{1+x}}{2}\right)^{n+1}}{\sqrt{1+x}}\\ &= 2^{-n}\sum_{j=0}^{\lceil\frac{n-1}{2}\rceil}\binom{n+1}{2j+1}(1+x)^j.\end{aligned} \tag{3.8}$$

From the left and right hand sides of (3.8) let $n=2m+1$ so that

$$\sum_{r=0}^{[(2m+1)/2]}\binom{2m+1-r}{r}\left(\frac{x}{4}\right)^r = 2^{-(2m+1)}\sum_{r=0}^{m}\binom{2m+2}{2r+1}(1+x)^r.$$

Expanding the right hand side and collecting powers of x we have

$$\sum_{r=0}^{[(2m+1)/2]}\binom{2m+1-r}{r}\left(\frac{x}{4}\right)^r = 2^{-(2m+1)}\sum_{r=0}^{m}x^r\left\{\sum_{j=r}^{m}\binom{2m+2}{2j+1}\binom{j}{r}\right\}$$

and equating coefficients of x we have the novel identity

$$\sum_{j=r}^{m}\binom{2m+2}{2j+1}\binom{j}{r} = 2^{2(m-r)+1}\binom{2m+1-r}{r}.$$

Now, to solve the problem of Krafft and Schaefer, add (3.4) and (3.5) so that

$$2^{-n}\sum_{j=0}^{\lceil\frac{n+1}{2}\rceil}\binom{n+1}{2j}(1+x)^j = \left(\frac{1+\sqrt{1+x}}{2}\right)^{n+1}+\left(\frac{1-\sqrt{1+x}}{2}\right)^{n+1}. \tag{3.9}$$

The ratio of (3.9) and (3.7), after putting $2m=n+1$ is

$$a_m = \frac{\sum_{j=0}^{m}\binom{2m}{2j}(1+x)^j}{\sum_{j=0}^{m-1}\binom{2m}{2j+1}(1+x)^j} = \sqrt{1+x}\left\{\frac{1+\left(\frac{1-\sqrt{1+x}}{1+\sqrt{1+x}}\right)^{2m}}{1-\left(\frac{1-\sqrt{1+x}}{1+\sqrt{1+x}}\right)^{2m}}\right\}$$

and $\lim_{m\to\infty} a_m = \sqrt{1+x}$, which solves Krafft and Schaefer's problem after replacing $1+x$ with x^*.

It is of some passing interest to note Wilf [10] derived the left hand side of (2.10) for $c=b$, and $R=1$. The derivation, by Wilf, counts the number of words of n letters over an alphabet of b letters that do not contain the substring of a word of a letters. There is another intriguing connection of the classical Fibonacci sequence with the Lambert series. The series $L(x) = \sum_{r=1}^{\infty} \frac{x^r}{1-x^r}$ was presented by Lambert circa 1771, and has been studied extensively. A closed form representation of the Lambert series may be useful, because of its possible importance in prime number theory. For the Fibonacci sequence (3.1) with $b=c=1$ it may be shown, see Knopp [6], that $\sum_{j=1}^{\infty} \frac{1}{f_{2j}} = \sqrt{5}\left\{L\left(\frac{3-\sqrt{5}}{2}\right) - L\left(\frac{7-3\sqrt{5}}{2}\right)\right\}$. Finally, from (2.1), for the case of $R=2$ and $a=1$ we require $b^2+4c \neq 0$ for (2.7) to have distinct zeros. From (2.9) and (2.11), we therefore obtain

$$\begin{aligned} &\sum_{r=0}^{n} (r+1) \binom{2n+1-r}{r+1} c^r b^{2(n-r)} \\ &= \sum_{j=0}^{1} \sum_{\mu=0}^{1} Q_{2,\mu}(\xi_j) \binom{2n+1}{1-\mu} \xi_j^{2n+\mu}, \end{aligned} \tag{3.10}$$

where, from using (2.3)

$$Q_{2,0}(\xi_j) = \frac{\xi_j^2}{(2\xi_j - b)^2}, \quad Q_{2,1}(\xi_j) = \frac{2\xi_j(\xi_j - b)}{(2\xi_j - b)^3}, \; j=0,1 \tag{3.11}$$

and

$$2\xi_0 = b + \sqrt{b^2+4c}, \quad 2\xi_1 = b - \sqrt{b^2+4c}$$

are the two zeros of (2.7).

If $b^2+4c>0$, ξ_j will be real zeros, and

if $b^2+4c<0$, ξ_j will be complex conjugate zeros. However, the sum (3.10) will be a real number.

From (3.10) and (3.11) we have that

$$\begin{aligned} &\sum_{r=0}^{n} (r+1) \binom{2n+1-r}{r+1} c^r b^{2(n-r)} \\ &= \frac{(2n+1)}{(b^2+4c)} \left[\xi_0^{2n+2} + \xi_1^{2n+2}\right] + \frac{2}{(b^2+4c)^{\frac{3}{2}}} \left[(\xi_0 - b)\,\xi_0^{2n+2} - (\xi_1 - b)\,\xi_1^{2n+2}\right]. \end{aligned}$$

In a follow up paper we will investigate the case of multiple zeros of the characteristic function (2.7).

4. Conclusion.

We have shown that many identities of binomial type sums may be generated by an application of the Z transform. The infinite sum of (2.10) may be treated in a similar fashion as described in this paper and details for $R=1$ may be seen in the paper by Sofo and Cerone[8]. The 'continuous' version of the infinite sum (2.10) may also be seen in the paper by Sofo and Cerone[9].

References

[1] Binz, J. C. *Closed form of two sums.* Problem 1498 and solution by J. Rosenberg. Mathematics Magazine, **Vol. 70, 1997, pp.201-214**.

[2] Hendel, R. J., and Cooke, C. K. *Recursive properties of trigonometric products.* Applications of Fibonacci Numbers, **Vol. 6, 1996, pp. 201- 214**. Editors G. E. Bergum, A. N. Philippou and A. F. Horadam. Kluger Academic Publishers, Netherlands.

[3] Hilton, P., and Pederson, J. *A Fresh Look at Old Favourites: The Fibonacci and Lucas Sequences Revisited.* Australian Mathematical Society Gazette, **Vol. 25, No.3, pp.146-160, 1998.**

[4] Horadam, A. F. *Jacobsthal Representation Numbers.* The Fibonacci Quarterly, **Vol. 34, pp.40-53, 1996.**

[5] Horadam, A. F. *Jacobsthal Polynomials.* The Fibonacci Quarterly, **Vol. 35, No. 2, pp.137-149, 1997.**

[6] Knopp, K. *Theory and Applications of Infinite Series.* Blackie and Son, London, $\mathbf{2}^{nd}$ **Edition**, **1963**.

[7] Krafft, O., and Schaefer, M. American Mathematical Monthly, **Vol. 104, p.871, Prob. No.10625, 1997.**

[8] Sofo, A., and Cerone, P. *On a Fibonacci Related Series.* The Fibonacci Quarterly, **Vol. 36, pp. 211-215, June-July 1998.**

[9] Sofo, A., and Cerone, P. *Generalisation of Euler's Identity.* Bull. Austral. Math. Soc., **Vol. 58, pp. 359-371, 1998.**

[10] Wilf, H. S. *Generatingfunctionology.* Academic Press Inc. New York, **1994**.

School of Communications and Informatics, Victoria University of Technology, PO Box 14428 Melbourne City MC, Victoria 8001, Australia