A GENERAL OSTROWSKI TYPE INEQUALITY FOR DOUBLE INTEGRALS

G. HANNA, S.S. DRAGOMIR, AND P. CERONE

ABSTRACT. Some generalisations of an Ostrowski Type Inequality in two dimensions for $n-$time differentiable mappings are given. The result is an Integral Inequality with bounded $n-$time derivatives. This is employed to approximate double integrals using one dimensional integrals and function evaluations at the boundary and interior points.

1. INTRODUCTION

The classical Ostrowski Integral Inequality (see [2, p. 468]) in one dimension stipulates a bound between a function evaluated at an interior point x and the average of the function f over an interval. That is,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty \tag{1.1}$$

for all $x\in[a,b]$, where $f'\in L_\infty(a,b)$ and $f:[a,b]\to\mathbb{R}$ is a differentiable mapping on (a,b).

Here, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant. We also observe that the tightest bound is obtained at $x=\frac{a+b}{2}$, resulting in the well-known mid-point inequality. In [1], P. Cerone, S.S. Dragomir and J. Roumeliotis proved the following Ostrowski type inequality for $n-$time differentiable mappings.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)}\in L_\infty[a,b]$. Then for all $x\in[a,b]$, we have the inequality:*

$$\begin{aligned}\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| &\leq \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right]\\ &\leq \frac{\left\|f^{(n)}\right\|_\infty(b-a)^{n+1}}{(n+1)!}\end{aligned} \tag{1.2}$$

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where $\left\|f^{(n)}\right\|_{\infty} := \sup_{t\in[a,b]} \left|f^{(n)}(t)\right| < \infty.$

For other similar results for $n-$time differentiable mappings, see the paper [7] by Fink and [8] by Anastassiou.

In [3] and [4] the authors proved some inequalities of Ostrowski type for double integrals in terms of different norms.

In this paper we combine the above two results and develop them in two dimensions to obtain a generalization of the Ostrowski inequality for n-time differentiable mappings using different types of norms.

The result presented here approximates a two-dimensional integral for $n-$time differentiable mappings via the application of function evaluations of one dimensional integrals at the boundary and an interior point.

2. Integral Identities

The following result holds.

Theorem 2. *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be a continuous mapping such that the following partial derivatives* $\frac{\partial^{l+k} f(\cdot,\cdot)}{\partial x^k \partial y^l}$, $k = 0,1,...,n-1$, $l = 0,1,...,m-1$ *exist and are continuous on* $[a,b]\times[c,d]$. *Further, for* $K_n : [a,b]^2 \to \mathbb{R}$, $S_m : [c,d]^2 \to \mathbb{R}$ *given by*

$$\begin{cases} K_n(x,t) := \begin{cases} \frac{(t-a)^n}{n!}, & t\in[a,x] \\ \frac{(t-b)^n}{n!}, & t\in(x,b] \end{cases} \\ S_m(y,s) := \begin{cases} \frac{(s-c)^m}{m!}, & s\in[c,y] \\ \frac{(s-d)^m}{m!}, & s\in(y,d] \end{cases} \end{cases} \tag{2.1}$$

then for all $(x,y)\in[a,b]\times[c,d]$, *we have the identity:*

$$\begin{aligned} &\int_a^b \int_c^d f(t,s)\,ds\,dt \\ =\ & \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} X_k(x)Y_l(y)\frac{\partial^{l+k} f(x,y)}{\partial x^k \partial y^l} \\ &+(-1)^m \sum_{k=0}^{n-1} X_k(x)\int_c^d S_m(y,s)\frac{\partial^{k+m} f(x,s)}{\partial x^k \partial s^m}ds \\ &+(-1)^n \sum_{l=0}^{m-1} Y_l(y)\int_a^b K_n(x,t)\frac{\partial^{n+l} f(t,y)}{\partial t^n \partial y^l}dt \\ &+(-1)^{m+n}\int_a^b\int_c^d K_n(x,t)S_m(y,s)\frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m}ds\,dt, \end{aligned} \tag{2.2}$$

where

$$\begin{cases} X_k(x) = \frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}, \\ Y_l(y) = \frac{(d-y)^{l+1}+(-1)^l(y-c)^{l+1}}{(l+1)!}. \end{cases} \tag{2.3}$$

Proof. Applying the identity (see [1])

$$\int_a^b g(t)\,dt = \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]g^{(k)}(x) + (-1)^n\int_a^b P_n(x,t)\,g^{(n)}(t)\,dt, \tag{I}$$

where

$$P_n(x,t)=\begin{cases}\frac{(t-a)^n}{n!} & \text{if } \ t\in[a,x],\\[2mm] \frac{(t-b)^n}{n!} & \text{if } \ t\in(x,b],\end{cases}$$

which has been used essentially in the proof of Theorem 1, for the partial mapping $f(\cdot,s)$, $s\in[c,d]$, we can write

$$\int_a^b f(t,s)\,dt = \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]\frac{\partial^k f(x,s)}{\partial x^k} + (-1)^n\int_a^b K_n(x,t)\,\frac{\partial^n f(t,s)}{\partial t^n}dt \tag{2.4}$$

for every $x\in[a,b]$ and $s\in[c,d]$.

Integrating (2.4) over s on $[c,d]$, we deduce

$$\begin{aligned}&\int_a^b\int_c^d f(t,s)\,ds\,dt \\ &= \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]\int_c^d\frac{\partial^k f(x,s)}{\partial x^k}ds \\ &\quad + (-1)^n\int_a^b K_n(x,t)\left(\int_c^d\frac{\partial^n f(t,s)}{\partial t^n}ds\right)dt\end{aligned} \tag{2.5}$$

for all $x\in[a,b]$.

Applying the identity (I) again for the partial mapping $\frac{\partial^k f(x,\cdot)}{\partial x^k}$ on $[c,d]$, we obtain

$$\begin{aligned}&\int_c^d\frac{\partial^k f(x,s)}{\partial x^k}ds \\ &= \sum_{l=0}^{m-1}\left[\frac{(d-y)^{l+1}+(-1)^l(y-c)^{l+1}}{(l+1)!}\right]\frac{\partial^l}{\partial y^l}\left(\frac{\partial^k f(x,y)}{\partial x^k}\right) \\ &\quad + (-1)^m\int_c^d S_m(y,s)\,\frac{\partial^m}{\partial s^m}\left(\frac{\partial^k f(x,s)}{\partial x^k}\right)ds \\ &= \sum_{l=0}^{m-1}\left[\frac{(d-y)^{l+1}+(-1)^l(y-c)^{l+1}}{(l+1)!}\right]\frac{\partial^{l+k}f(x,y)}{\partial x^k\partial y^l} \\ &\quad + (-1)^m\int_c^d S_m(y,s)\,\frac{\partial^{k+m}f(x,s)}{\partial x^k\partial s^m}ds.\end{aligned} \tag{2.6}$$

In addition, the identity (2.4) applied for the partial derivative $\frac{\partial^n f(t,\cdot)}{\partial t^n}$ also gives

$$\int_c^d \frac{\partial^n (t,s)}{\partial t^n} ds = \sum_{l=0}^{m-1} \left[\frac{(d-y)^{l+1} + (-1)^l (y-c)^{l+1}}{(l+1)!} \right] \frac{\partial^{n+l} f(t,y)}{\partial t^n \partial y^l} + (-1)^m \int_c^d S_m(y,s) \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} ds. \tag{2.7}$$

Using (2.6) and (2.7) and substituting into (2.5) will produce the result (2.2), and thus the theorem is proved. ∎

Corollary 1. *With the assumptions as in Theorem 2, we have the representation*

$$\begin{aligned} &\int_a^b \int_c^d f(t,s)\, ds\, dt \\ &= \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} X_k\left(\frac{a+b}{2}\right) Y_l\left(\frac{c+d}{2}\right) \frac{\partial^{l+k} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)}{\partial x^k \partial y^l} \\ &\quad + (-1)^m \sum_{k=0}^{n-1} X_k\left(\frac{a+b}{2}\right) \int_c^d \tilde{S}_m(s) \frac{\partial^{k+m} f\left(\frac{a+b}{2}, s\right)}{\partial x^k \partial s^m} ds \\ &\quad + (-1)^n \sum_{l=0}^{m-1} Y_l\left(\frac{c+d}{2}\right) \int_a^b \tilde{K}_n(t) \frac{\partial^{n+l} f\left(t, \frac{c+d}{2}\right)}{\partial t^n \partial y^l} dt \\ &\quad + (-1)^{m+n} \int_a^b \int_c^d \tilde{K}_n(t) \tilde{S}_m(s) \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} ds\, dt, \end{aligned} \tag{2.8}$$

where $X_k(\cdot)$ and $Y_l(\cdot)$ are as given in (2.2) and so

$$X_k\left(\frac{a+b}{2}\right) = \left[\frac{1+(-1)^k}{(k+1)!}\right] \frac{(b-a)^{k+1}}{2^{k+1}},$$

$$Y_l\left(\frac{c+d}{2}\right) = \left[\frac{1+(-1)^l}{(l+1)!}\right] \frac{(d-c)^{l+1}}{2^{l+1}},$$

and $\tilde{K}_n : [a,b] \to \mathbb{R}$, $\tilde{S}_m : [c,d] \to \mathbb{R}$ are given by

$$\tilde{K}_n(t) = K_n\left(\frac{a+b}{2}, t\right)$$

and

$$\tilde{S}_m(s) = S_m\left(\frac{c+d}{2}, s\right)$$

on using (2.1).

Corollary 2. *Let f be as in Theorem 2. Then we have the following identity*

$$\int_a^b \int_c^d f(t,s)\,ds\,dt \tag{2.9}$$

$$= \frac{1}{4}\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\left[\frac{(b-a)^{k+1}}{(k+1)!}\right]\times\left[\frac{(d-c)^{l+1}}{(l+1)!}\right]$$

$$\times\frac{\partial^{l+k}}{\partial x^k \partial y^l}\left[f(a,c)+(-1)^l f(a,d)+(-1)^k f(b,c)+(-1)^{l+k} f(b,d)\right]$$

$$+\frac{1}{4}(-1)^m\sum_{k=0}^{n-1}\left[\frac{(b-a)^{k+1}}{(k+1)!}\right]$$

$$\times\left[\int_c^d Y_{m-1}(s)\frac{\partial^{k+m}}{\partial x^k \partial s^m}\left[f(a,s)+(-1)^k f(b,s)\right]ds\right]$$

$$+\frac{1}{4}(-1)^n\sum_{l=0}^{m-1}\left[\frac{(d-c)^{l+1}}{(l+1)!}\right]\times\left[\int_a^b X_{n-1}(t)\frac{\partial^{n+l}}{\partial t^n \partial y^l}\left[f(t,c)+(-1)^k f(t,d)\right]dt\right]$$

$$+\frac{1}{4}\int_a^b\int_c^d X_{n-1}(t)\cdot Y_{m-1}(s)\frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m}ds\,dt,$$

where $X_{n-1}(t)$ and $Y_{m-1}(s)$ are as given by (2.3).

Proof. By substituting $(x,y)=(a,c),(a,d),(b,c),(b,d)$ respectively and summing the resulting identities and after some simplification, we get the desired inequality (2.9). ∎

3. Some Integral Inequalities

We start with the following result

Theorem 3. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be continuous on $[a,b]\times[c,d]$, and assume that $\frac{\partial^{n+m} f}{\partial t^n \partial s^m}$ exist on $(a,b)\times(c,d)$. Then we have the inequality*

$$\Bigg|\int_a^b\int_c^d f(t,s)\,ds\,dt-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}X_k(x)\cdot Y_l(y)\frac{\partial^{l+k} f(x,y)}{\partial x^k \partial y^l} \tag{3.1}$$

$$-(-1)^m\sum_{k=0}^{n-1}X_k(x)\int_c^d S(y,s)\frac{\partial^{k+m} f(x,s)}{\partial x^k \partial s^m}ds$$

$$-(-1)^n\sum_{l=0}^{m-1}Y_l(y)\int_a^b K(x,t)\frac{\partial^{n+l} f(t,y)}{\partial t^n \partial y^l}dt\Bigg|$$

$$\leq \begin{cases} \frac{1}{(n+1)!(m+1)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right]\times\left[(y-c)^{m+1}+(d-y)^{m+1}\right]\times\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty \\ \text{if } \frac{\partial^{n+m}f}{\partial t^n\partial s^m}\in L_\infty([a,b]\times[c,d]); \\ \frac{1}{n!m!}\left[\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{nq+1}\right]^{\frac{1}{q}}\times\left[\frac{(y-c)^{mq+1}+(d-y)^{mq+1}}{mq+1}\right]^{\frac{1}{q}}\times\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_p \\ \text{if } \frac{\partial^{n+m}f}{\partial t^n\partial s^m}\in L_p([a,b]\times[c,d]),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{4n!m!}\left[(x-a)^n+(b-x)^n+\left|(x-a)^n-(b-x)^n\right|\right] \\ \quad\times\left[(y-c)^m+(d-y)^m+\left|(y-c)^m-(d-y)^m\right|\right]\times\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_1 \\ \text{if } \frac{\partial^{n+m}f}{\partial t^n\partial s^m}\in L_1([a,b]\times[c,d]) \end{cases}$$

for all $(x,y)\in[a,b]\times[c,d]$, *where*

$$\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty = \sup_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|<\infty,$$

$$\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_p = \left(\int_a^b\int_c^d\left|\frac{\partial^{n+m}}{\partial t^n\partial s^m}f(t,s)\right|^p dtds\right)^{\frac{1}{p}}<\infty.$$

Proof. Using Theorem 2, we get from (2.2)

$$\begin{aligned} &\left|\int_a^b\int_c^d f(t,s)\,ds\,dt-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}X_k(x)\cdot Y_l(y)\frac{\partial^{l+k}f(x,y)}{\partial x^k\partial y^l}\right. \\ &-(-1)^m\sum_{k=0}^{n-1}X_k(x)\int_c^d S(y,s)\frac{\partial^{k+m}f(x,s)}{\partial x^k\partial s^m}ds \\ &\left.-(-1)^n\sum_{l=0}^{m-1}Y_l(y)\int_a^b K(x,t)\frac{\partial^{n+l}f(t,y)}{\partial t^n\partial y^l}dt\right| \\ =&\left|\int_a^b\int_c^d K_n(x,t)\,S_m(y,s)\,\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^n}ds\,dt\right| \\ \leq&\int_a^b\int_c^d\left|K_n(x,t)\,S_m(y,s)\right|\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^n}\right|ds\,dt. \end{aligned}\tag{3.2}$$

Using Hölder's inequality and properties of the modulus and integral, then we have that

$$\int_a^b\int_c^d\left|K_n(x,t)\,S_m(y,s)\right|\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|ds\,dt\tag{3.3}$$

$$\leq \begin{cases} \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty \int_a^b \int_c^d |K_n(x,t) S_m(y,s)| \, dt \, ds \\ \\ \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p \left(\int_a^b \int_c^d |K_n(x,t) S_m(y,s)|^q \, dt \, ds \right)^{\frac{1}{q}}, \\ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_1 \sup_{(t,s) \in [a,b] \times [c,d]} |K_n(x,t) S_m(y,s)| . \end{cases}$$

Now, from (3.3) and using (2.1),

$$\begin{aligned} & \int_a^b \int_c^d |K_n(x,t) S_m(y,s)| \, dt \, ds \\ = & \int_a^b |K_n(x,t)| \, dt \int_c^d |S_m(y,s)| \, ds \\ = & \left[\int_a^x \frac{(t-a)^n}{n!} dt + \int_x^b \frac{(b-t)^n}{n!} dt \right] \times \left[\int_c^y \frac{(s-c)^m}{m!} ds + \int_y^d \frac{(d-s)^m}{m!} ds \right] \\ = & \frac{\left[(x-a)^{n+1} + (b-x)^{n+1} \right] \left[(y-c)^{m+1} + (d-y)^{m+1} \right]}{(n+1)! \, (m+1)!} \end{aligned}$$

giving the first inequality in (3.1).

Further, on using (2.1) and from (3.3)

$$\begin{aligned} & \left(\int_a^b \int_c^d |K_n(x,t) S_m(y,s)|^q \, ds \, dt \right)^{\frac{1}{q}} \\ = & \left(\int_a^b |K_n(x,t)|^q \, dt \right)^{\frac{1}{q}} \left(\int_c^d |S_m(y,s)|^q \, ds \, dt \right)^{\frac{1}{q}} \\ = & \frac{1}{n!m!} \left[\int_a^x (t-a)^{nq} \, dt + \int_x^b (b-t)^{nq} \, dt \right]^{\frac{1}{q}} \\ & \times \left[\int_c^y (s-c)^{mq} \, ds + \int_y^d (d-s)^{mq} \, ds \right]^{\frac{1}{q}} \\ = & \frac{1}{n!m!} \left[\frac{(x-a)^{nq+1} + (b-x)^{nq+1}}{nq+1} \right]^{\frac{1}{q}} \times \left[\frac{(y-c)^{mq+1} + (d-y)^{mq+1}}{mq+1} \right]^{\frac{1}{q}} \end{aligned}$$

producing the second inequality in (3.1).

Finally, from (2.1) and (3.3),

$$\begin{aligned} & \sup_{(t,s) \in [a,b] \times [c,d]} |K_n(x,t) S_m(y,s)| \\ = & \sup_{t \in [a,b]} |K_n(x,t)| \sup_{s \in [c,d]} |S_m(y,s)| \\ = & \max \left\{ \frac{(x-a)^n}{n!}, \frac{(b-x)^n}{n!} \right\} \times \max \left\{ \frac{(y-c)^m}{m!}, \frac{(d-y)^m}{m!} \right\} \end{aligned}$$

$$= \frac{1}{n!m!}\left[\frac{(x-a)^n+(b-x)^n}{2}+\left|\frac{(x-a)^n-(b-x)^n}{2}\right|\right] \times\left[\frac{(y-c)^m+(d-y)^m}{2}+\left|\frac{(y-c)^m+(d-y)^m}{2}\right|\right].$$

gives the inequality in (3.1) where we have used the fact that

$$\max\{X,Y\}=\frac{X+Y}{2}+\left|\frac{Y-X}{2}\right|.$$

Thus the theorem is now completely proved. ∎

¿From the results of Theorem 3 above, we have the following corollary.

Corollary 3. *With the assumptions of Theorem 3, we have the inequality*

$$\begin{aligned}(3.4)\quad &\Bigg|\int_a^b\int_c^d f(t,s)\,ds\,dt \\ &-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}X_k\left(\frac{a+b}{2}\right)Y_l\left(\frac{c+d}{2}\right)\frac{\partial^{l+k}}{\partial x^k\partial y^l}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ &-(-1)^m\sum_{k=0}^{n-1}X_k\left(\frac{a+b}{2}\right)\int_c^d\tilde{S}_m(s)\frac{\partial^{k+m}}{\partial x^k\partial s^m}f\left(\frac{a+b}{2},s\right)ds \\ &-(-1)^n\sum_{l=0}^{m-1}Y_l\left(\frac{c+d}{2}\right)\int_a^b\tilde{K}_n(t)\frac{\partial^{n+l}}{\partial t^n\partial y^l}f\left(t,\frac{c+d}{2}\right)dt\Bigg| \\ \leq &\begin{cases}\frac{1}{2^{n+m}(n+1)!(m+1)!}(b-a)^{n+1}(d-c)^{m+1}\times\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty; \\ \frac{1}{2^{n+m}n!m!}\left[\frac{(b-a)^{nq+1}(d-c)^{mq+1}}{(nq+1)(mq+1)}\right]^{\frac{1}{q}}\times\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_p; \\ \frac{1}{2^{n+m}n!m!}(b-a)^n(d-c)^m\times\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_1,\end{cases}\end{aligned}$$

where $\|\cdot\|_p$ $(p\in[1,\infty])$ *are the Lebesgue norms on* $[a,b]\times[c,d]$.

Proof. Taking $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in (3.1) readily produces the result as stated. ∎

These are the tightest possible for their respective *Lebesgue* norms, because of the symmetric and convex nature of the bounds in (3.1).

Remark 1. *For* $n=m=1$ *in (3.4) and* $\frac{\partial^2 f}{\partial t\partial s}$ *belonging to the appropriate Lebesgue spaces on* $[a,b]\times[c,d]$*, we have*

$$\begin{aligned}(3.5)\quad &\Bigg|\int_a^b\int_c^d f(t,s)\,ds\,dt-(b-a)(d-c)f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ &+(b-a)\int_c^d\tilde{S}_1(s)\frac{\partial}{\partial s}f\left(\frac{a+b}{2},s\right)ds+(d-c)\int_a^b\tilde{K}_1(t)\frac{\partial}{\partial t}f\left(t,\frac{c+d}{2}\right)dt\Bigg|\end{aligned}$$

$$\leq \begin{cases} \frac{1}{16}(b-a)^2(d-c)^2 \times \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty ; \\ \frac{1}{4}\left[\frac{(b-a)^{q+1}(d-c)^{q+1}}{(q+1)^2} \right]^{\frac{1}{q}} \times \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_p ; \\ \frac{1}{4}(b-a)(d-c) \times \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_1 , \end{cases}$$

and thus some of the results of [5] *and* [6] *are recaptured.*

Corollary 4. *With the assumptions on f as outlined in Theorem 3, we can obtain another result which is a generalization of the Trapezoid inequality*

$$(3.6) \qquad \left| \int_a^b \int_c^d f(t,s)\, ds\, dt - \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \frac{(b-a)^{k+1}}{(k+1)!} \cdot \frac{(d-c)^{l+1}}{(l+1)!} \right.$$

$$\times \left[\frac{f(a,c) + (-1)^l f(a,d) + (-1)^k f(b,c) + (-1)^{k+l} f(b,d)}{4} \right]$$

$$-(-1)^m \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \int_c^d Y_l(s) \frac{\partial^{k+m}}{\partial x^k \partial s^m} \left[\frac{f(a,s) + (-1)^k f(b,s)}{4} \right] ds$$

$$\left. -(-1)^n \sum_{l=0}^{m-1} \left[\frac{(d-c)^{l+1}}{(l+1)!} \right] \int_a^b X_k(t) \frac{\partial^{l+n}}{\partial t^n \partial y^l} \left[\frac{f(t,c) + (-1)^l f(t,d)}{4} \right] dt \right|$$

$$\leq \begin{cases} \kappa_{n,m} \frac{(b-a)^{n+1}(d-c)^{m+1}}{(n+1)!(m+1)!} \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty ; \\ \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_\infty([a,b] \times [c,d]); \\ \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p \left(\int_a^b |T_n(a,b;t)|^q dt \right)^{\frac{1}{q}} \left(\int_c^d |T_m(c,d;s)|^q ds \right)^{\frac{1}{q}} \\ \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_p([a,b] \times [c,d]),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^n (d-c)^m}{4 n! m!} \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_1 , \\ \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_1([a,b] \times [c,d]). \end{cases}$$

where

$$\kappa_{n,m} := \begin{cases} 1 & \text{if} \quad n = 2r_1 \text{ and } m = 2r_2 \,, \\ \frac{2^n-1}{2^n} & \text{if} \quad n = 2r_1+1 \text{ and } m = 2r_2 \,, \\ \frac{2^m-1}{2^m} & \text{if} \quad n = 2r_1 \text{ and } m = 2r_2+1 \,, \\ \frac{(2^n-1)}{2^n} \cdot \frac{(2^m-1)}{2^m} & \text{if} \quad n = 2r_1+1 \text{ and } m = 2r_2+1 \end{cases}$$

Proof. Using the identity (2.9), we find that

$$\begin{aligned} & \left| \int_a^b \int_c^d f(t,s)\,ds\,dt - \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \frac{(b-a)^{k+1}}{(k+1)!} \cdot \frac{(d-c)^{l+1}}{(l+1)!} \right. \\ & \times \frac{\partial^{l+k}}{\partial x^k \partial y^l} \left[\frac{f(a,c) + (-1)^l f(a,d) + (-1)^k f(b,c) + (-1)^{k+l} f(b,d)}{4} \right] \\ & - (-1)^m \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \int_c^d \frac{(s-c)^m + (s-d)^m}{m!} \\ & \times \frac{\partial^{k+m}}{\partial x^k \partial s^m} \left[\frac{f(a,s) + (-1)^k f(b,s)}{4} \right] ds \\ & \quad - (-1)^n \sum_{l=0}^{m-1} \left[\frac{(d-c)^{l+1}}{(l+1)!} \right] \int_a^b \frac{(t-a)^n + (t-b)^n}{n!} \\ & \quad \left. \times \frac{\partial^{n+l}}{\partial y^l \partial t^n} \left[\frac{f(t,c) + (-1)^l f(t,d)}{4} \right] dt \right| \\ = & \quad \left| \int_a^b \int_c^d T_n(a,b;t)\, T_m(c,d;s) \frac{\partial^{n+m} f}{\partial t^n \partial s^m} dsdt \right| \\ \leq & \begin{cases} \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty \int_a^b \int_c^d |T_n(a,b;t)\, T_m(c,d;s)|\, dt\, ds \\ \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_\infty([a,b]\times[c,d]); \\ \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p \left(\int_a^b |T_n(a,b;t)|^q\, dt \right)^{\frac{1}{q}} \left(\int_c^d |T_m(c,d;s)|^q\, ds \right)^{\frac{1}{q}} \\ \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_p([a,b]\times[c,d]),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_1 \sup\limits_{(t,s)\in[a,b]\times[c,d]} |T_n(a,b;t)\, T_m(c,d;s)| \\ \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_1([a,b]\times[c,d]). \end{cases} \end{aligned}$$

where

$$\begin{aligned} T_n(a,b;t) &= \frac{1}{2}\left[\frac{(b-t)^n+(-1)^n(t-a)^n}{n!}\right], \\ T_m(c,d;s) &= \frac{1}{2}\left[\frac{(d-s)^m+(-1)^m(s-c)^m}{m!}\right]. \end{aligned}$$

Now consider $\int_a^b |T_n(a,b;t)|\,dt$. As may be seen, explicit evaluation of the integral depends on whether n is even or odd.

(i) If n is even, put $n=2r_1$. Therefore,

$$\begin{aligned} \int_a^b |T_n(a,b;t)|\,dt &= \frac{1}{(2r_1)!}\int_a^b \frac{(b-t)^{2r_1}+(t-a)^{2r_1}}{2}dt \\ &= \frac{1}{(2r_1)!}\cdot\frac{1}{2}\left[\frac{(b-a)^{2r_1+1}}{2r_1+1}+\frac{(b-a)^{2r_1+1}}{2r_1+1}\right] \\ &= \frac{(b-a)^{2r_1+1}}{(2r_1+1)!}=\frac{(b-a)^{n+1}}{(n+1)!}. \end{aligned}$$

Similarly,

$$\int_c^d |T_m(c,d;s)|\,ds=\frac{1}{(2r_2)!}\int_c^d\frac{(d-s)^{2r_2}+(s-c)^{2r_2}}{2}ds=\frac{(d-c)^{m+1}}{(m+1)!}.$$

(ii) Now, if n is odd, that is, $n=2r_1+1$, then

$$T_n(a,b;t) = \frac{(b-t)^{2r_1+1}-(t-a)^{2r_1+1}}{2\,(2r_1+1)!}.$$

Let $g(t)=(b-t)^{2r_1+1}-(t-a)^{2r_1+1}$.
We can observe that

$$\begin{cases} g(t)<0 \text{ for all } t\in(\frac{a+b}{2},b] \\ g(t)=0 \quad \text{at } t=\frac{a+b}{2} \\ g(t)>0 \text{ for all } t\in[a,\frac{a+b}{2}). \end{cases}$$

Thus

$$\begin{aligned} &2(2r_1+1)!\int_a^b |T_n(a,b;t)|\,dt \\ =& \left[\int_a^{\frac{a+b}{2}}\left[(b-t)^{2r_1+1}-(t-a)^{2r_1+1}\right]dt\right. \\ &\left.+\int_{\frac{a+b}{2}}^b\left[(t-a)^{2r_1+1}-(b-t)^{2r_1+1}\right]dt\right] \\ =& \left[2\cdot\frac{(b-a)^{2r_1+2}}{2r_1+2}-4\frac{\left(\frac{b-a}{2}\right)^{2r_1+2}}{2r_1+2}\right] \end{aligned}$$

and so

$$\begin{aligned}\int_a^b |T_n(a,b;t)|\,dt &= \frac{(b-a)^{2r_1+2}}{(2r_1+2)(2r_1+1)!}\left[1-\frac{1}{2^{2r_1+1}}\right] \\ &= \frac{(b-a)^{2r_1+2}}{(2r_1+2)!}\left[\frac{2^{2r_1+1}-1}{2^{2r_1+1}}\right] = \frac{(b-a)^{n+1}}{(n+1)!}\left[\frac{2^n-1}{2^n}\right].\end{aligned}$$

Similarly,

$$\int_c^d |T_m(c,d;s)|\,ds = \frac{(d-c)^{m+1}}{(m+1)!}\left[\frac{2^m-1}{2^m}\right].$$

and this gives the first inequality in (3.6).

Now, for the third inequality we have,

$$\sup_{t\in[a,b]}|T_n(a,b;t)| = \frac{1}{2n!}\times\begin{cases}\sup\limits_{t\in[a,b]}\left((b-t)^n+(t-a)^n\right) = \frac{(b-a)^n}{2n!} & \text{for all } n \textit{ even} \\ \sup\limits_{t\in[a,b]}\left|(b-t)^n-(t-a)^n\right| = \frac{(b-a)^n}{2n!} & \text{for all } n \textit{ odd}\end{cases}$$

and this gives last part of the inequality in (3.6). The corollary is thus completely proved. ∎

Remark 2. *For $n=m=1$, we have that*

$$\begin{aligned}&\Bigg|\int_a^b\int_c^d f(t,s)\,ds\,dt + \frac{(b-a)(d-c)}{4}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right] \\ &-\frac{b-a}{2}\left[\int_c^d (f(a,s)+f(b,s))\,ds\right] - \frac{d-c}{2}\left[\int_a^b (f(t,c)+f(t,d))\,dt\right]\Bigg| \\ &\leq\begin{cases}\frac{(b-a)^2(d-c)^2}{4}\left[(x-a)^2+(b-x)^2\right]\left[(y-c)^2+(d-y)^2\right]\times\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty \\ \frac{1}{4}\left[\frac{((b-a)(d-c))^{q+1}}{(q+1)^2}\right]^{\frac{1}{q}}\times\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_p,\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)(d-c)}{4}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_1.\end{cases}\end{aligned}$$

Again, the same result was obtained by G. Hanna et al. in [5] *and S. Dragomir et al. in* [6].

4. Applications to Numerical Integration.

The following application in Numerical Integration is natural to be considered.

Theorem 4. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be as in Theorem 3. In addition, let I_v and J_μ be arbitrary divisions of $[a,b]$ and $[c,d]$ respectively, that is,*

$$I_v : a=\xi_0<\xi_1<\ldots<\xi_\nu=b,$$

where $x_i\in(\xi_i,\xi_{i+1})$ for $i=0,1,\ldots,\nu-1$, and

$$J_\mu : c=\tau_0<\tau_1<\ldots<\tau_\mu=d,$$

with $y_j \in (\tau_j, \tau_{j+1})$ *for* $j = 0, 1, ..., \mu - 1$, *then we have the cubature formula*

$$\int_a^b \int_c^d f(t,s)\, ds\, dt = \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \sum_{i=0}^{\nu-1} \sum_{j=0}^{\mu-1} X_k^{(i)}(x_i) Y_l^{(j)}(y_j) \frac{\partial^{i+j} f(x_i, y_j)}{\partial x^i \partial y^j} \tag{4.1}$$

$$+ (-1)^m \sum_{k=0}^{n-1} \sum_{i=0}^{\nu-1} \sum_{j=0}^{\mu-1} X_k^{(i)}(x_i) \int_{\tau_j}^{\tau_{j+1}} S_m^{(j)}(y_j, s) \frac{\partial^{k+m} f(x_i, s)}{\partial x^k \partial s^m} ds$$

$$+ (-1)^n \sum_{l=0}^{m-1} \sum_{i=0}^{\nu-1} \sum_{j=0}^{\mu-1} Y_l^{(j)}(y_j) \int_{\xi_i}^{\xi_{i+1}} K_n^{(i)}(x_i, t) \frac{\partial^{n+l} f(t, y_j)}{\partial t^n \partial y^l} dt$$

$$+ R(f, I_v, J_\mu, x, y),$$

where the remainder term satisfies the condition

$$|R(f, I_n, J_m, x, y)|$$

$$\leq \begin{cases} \frac{\left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty}{(n+1)!(m+1)!} \times \sum\limits_{i=0}^{\nu-1} \left[(x_i - \xi_i)^{n+1} + (\xi_{i+1} - x_i)^{n+1} \right] \\ \qquad \times \sum\limits_{j=0}^{\mu-1} \left[(y_j - \tau_j)^{m+1} + (\tau_{j+1} - y_j)^{m+1} \right] \\ \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_\infty([a,b] \times [c,d]); \\ \frac{\left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p}{n!m!(nq+1)^{\frac{2}{q}}} \times \sum\limits_{i=0}^{\nu-1} \left[(x_i - \xi_i)^{nq+1} + (\xi_{i+1} - x_i)^{nq+1} \right]^{\frac{1}{q}} \\ \qquad \times \sum\limits_{j=0}^{\mu-1} \left[(y_j - \tau_j)^{mq+1} + (\tau_{j+1} - y_j)^{mq+1} \right]^{\frac{1}{q}} \\ \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_p([a,b] \times [c,d]),\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{\left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_1}{4n!m!} \sum\limits_{i=0}^{\nu-1} \left[(x_i - \xi_i)^n + (\xi_{i+1} - x_i)^n + \left| (x_i - \xi_i)^n - (\xi_{i+1} - x_i)^n \right| \right] \\ \qquad \times \sum\limits_{j=0}^{\mu-1} \left[(y_j - \tau_j)^m + (\tau_{j+1} - y_j)^m + \left| (y_j - \tau_j)^m - (\tau_{j+1} - y_j)^m \right| \right] \\ \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_1([a,b] \times [c,d]); \end{cases}$$

where

$X_k^{(i)}(k = 0, 1, ...n - 1; i = 0, 1, ...\nu - 1)$, $Y_l^{(j)}(l = 0, 1, ...m - 1; j = 0, 1, ...\mu - 1)$
and

$K_n^{(i)}(i = 0, 1, ...\nu - 1)$, $S_m^{(j)}(j = 0, 1, ...\mu - 1)$ *are defined by*

$$X_k^{(i)}(x_i) := \frac{\left(\xi_{i+1}-x_i\right)^{k+1}+(-1)^k(x_i-\xi_i)^{k+1}}{(k+1)!},$$

$$Y_l^{(j)}(y_j) := \frac{(\tau_{j+1}-y_j)^{l+1}+(-1)^l(y_j-\tau_j)^{l+1}}{(l+1)!},$$

$$K_n^{(i)}(x_i,t) := \begin{cases} \frac{(t-\xi_i)^n}{n!} & , t\in[\xi_i, x_i] \\ \frac{\left(t-\xi_{i+1}\right)^n}{n!} & , t\in\left(x_i, \xi_{i+1}\right] \end{cases}$$

and

$$S_m^{(j)}(y_j,s) := \begin{cases} \frac{(s-\tau_j)^m}{m!} & , s\in[\tau_i, y_i] \\ \frac{(s-\tau_{j+1})^m}{m!} & , s\in(y_i, \tau_{j+1}] \end{cases}$$

The proof is obvious by Theorem 3 applied on the interval $\left[\xi_i, \xi_{i+1}\right]\times[\tau_j, \tau_{j+1}]$, $(i = 0, 1, ...\nu - 1; j = 0, 1, ...\mu - 1)$, and we omit the details.

Remark 3. *Similar result can be obtained if we use the other results obtained in section 3, but we omit the details.*

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.
E-mail address: georgey@sci.vu.edu.au

E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.vu.edu.au/SSDragomirWeb.html

E-mail address: pc@sci.vu.edu.au