

# GENERALISATIONS OF THE GRÜSS, CHEBYCHEV AND LUPAŞ INEQUALITIES FOR INTEGRALS OVER DIFFERENT INTERVALS

P. CERONE AND S.S. DRAGOMIR

ABSTRACT. Generalisations of the Grüss, Chebychev and Lupaş inequalities for integrals defined on two different intervals are given. Application in numerical analysis is demonstrated.

## 1. INTRODUCTION

For two measurable functions  $f, g : [a, b] \rightarrow \mathbb{R}$ , define the functional, which is known in the literature as Chebychev's functional

$$(1.1) \quad T(f, g; a, b) := \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{(b-a)^2} \int_a^b f(x) dx \cdot \int_a^b g(x) dx,$$

provided that the involved integrals exist.

The following inequality is well known in the literature as the Grüss inequality [9]

$$(1.2) \quad |T(f, g; a, b)| \leq \frac{1}{4} (M - m) (N - n),$$

provided that  $m \leq f \leq M$  and  $n \leq g \leq N$  a.e. on  $[a, b]$ , where  $m, M, n, N$  are real numbers. The constant  $\frac{1}{4}$  in (1.2) is the best possible.

Another inequality of this type is due to Chebychev (see for example [1, p. 207]). Namely, if  $f, g$  are absolutely continuous on  $[a, b]$  and  $f', g' \in L_\infty[a, b]$  and  $\|f'\|_\infty := \operatorname{ess\,sup}_{t \in [a, b]} |f'(t)|$ , then

$$(1.3) \quad |T(f, g; a, b)| \leq \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b-a)^2$$

and the constant  $\frac{1}{12}$  is the best possible.

Finally, let us recall a result by Lupaş (see for example [1, p. 210]), which states that:

$$(1.4) \quad |T(f, g; a, b)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a)^2,$$

provided  $f, g$  are absolutely continuous and  $f', g' \in L_2[a, b]$ . The constant  $\frac{1}{\pi^2}$  is the best possible here.

For other Grüss type inequalities, see the books [1] and [2], and the papers [3]-[10], where further references are given.

---

*Date:* March 21, 2000.

*1991 Mathematics Subject Classification.* Primary 26D15, 26D20, 26D99; Secondary 41A55.

*Key words and phrases.* Grüss, Chebychev and Lupaş inequalities, Numerical Analysis.

In this note, we point out generalizations of the above results for integrals defined on two different intervals  $[a, b]$  and  $[c, d]$  and demonstrate their application in numerical analysis.

## 2. THE RESULTS

We define the functional (generalised Chebychev functional)

$$(2.1) \quad T(f, g; a, b, c, d) \\ : = \frac{1}{b-a} \int_a^b f(x) g(x) dx + \frac{1}{d-c} \int_c^d f(y) g(y) dy \\ - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{d-c} \int_c^d g(y) dy - \frac{1}{b-a} \int_a^b g(x) dx \cdot \frac{1}{d-c} \int_c^d f(y) dy,$$

and the integral mean

$$(2.2) \quad M(f; a, b) := \frac{1}{b-a} \int_a^b f(x) dx.$$

The following result holds.

**Theorem 1.** *Let  $f, g : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be measurable on  $I$  and the intervals  $[a, b], [c, d] \subset I$ . Assume that the integrals involved in (2.1) exist. Then we have the inequality*

$$(2.3) \quad |T(f, g; a, b, c, d)| \\ \leq \left[ T(f; a, b) + T(f; c, d) + (M(f; a, b) - M(f; c, d))^2 \right]^{\frac{1}{2}} \\ \times \left[ T(g; a, b) + T(g; c, d) + (M(g; a, b) - M(g; c, d))^2 \right]^{\frac{1}{2}},$$

where

$$(2.4) \quad T(f; a, b) := \frac{1}{b-a} \int_a^b f^2(x) dx - \left( \frac{1}{b-a} \int_a^b f(x) dx \right)^2$$

and the integrals involved in the right membership of (2.3) exist.

*Proof.* A simple calculation shows that:

$$(2.5) \quad T(f, g; a, b, c, d) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d (f(x) - f(y))(g(x) - g(y)) dy dx.$$

This is a generalisation of the classical identity due to Korkine.

Using the Cauchy-Buniakowski-Schwartz inequality for double integrals, we have

$$(2.6) \quad |T(f, g; a, b, c, d)|^2 \\ = \left[ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d (f(x) - f(y))(g(x) - g(y)) dy dx \right]^2 \\ \leq \left[ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d (f(x) - f(y))^2 dy dx \right] \\ \times \left[ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d (g(x) - g(y))^2 dy dx \right] \\ = T(f, f; a, b, c, d) T(g, g; a, b, c, d).$$

We also remark that

$$\begin{aligned}
(2.7) \quad T(f, f; a, b, c, d) &= \frac{1}{b-a} \int_a^b f^2(x) dx + \frac{1}{d-c} \int_c^d f^2(y) dy \\
&\quad - 2 \cdot \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{d-c} \int_c^d f(y) dy \\
&= T(f; a, b) + T(f; c, d) + (M(f; a, b) - M(f; c, d))^2
\end{aligned}$$

and a similar identity holds for  $g$ .

Thus, using (2.6), we obtain the desired inequality (2.3). ■

**Remark 1.** If  $c = a$ ,  $d = b$ , then it may be noticed from (2.5) that

$$T(f, g; a, b, a, b) = 2T(f, g; a, b),$$

where  $T(f, g; a, b)$  is the classical Chebychev functional (1.1).

**Corollary 1.** Assume that  $f$  and  $g$  are as in Theorem 1 and, in addition

$$(2.8) \quad m_1 \leq f \leq M_1, \text{ a.e. on } [a, b] \text{ and } m_2 \leq f \leq M_2 \text{ a.e. on } [c, d],$$

$$(2.9) \quad n_1 \leq g \leq N_1, \text{ a.e. on } [a, b] \text{ and } n_2 \leq g \leq N_2 \text{ a.e. on } [c, d].$$

Then we have the inequality

$$\begin{aligned}
(2.10) \quad &|T(f, g; a, b, c, d)| \\
&\leq \left[ \frac{1}{4} (M_1 - m_1)^2 + \frac{1}{4} (M_2 - m_2)^2 + (M(f; a, b) - M(f; c, d))^2 \right]^{\frac{1}{2}} \\
&\quad \times \left[ \frac{1}{4} (N_1 - n_1)^2 + \frac{1}{4} (N_2 - n_2)^2 + (M(g; a, b) - M(g; c, d))^2 \right]^{\frac{1}{2}}.
\end{aligned}$$

*Proof.* The proof follows by (2.3), taking into account that, by the Grüss inequality (1.2),

$$\begin{aligned}
T(f; a, b) &\leq \frac{1}{4} (M_1 - m_1)^2, \\
T(f; c, d) &\leq \frac{1}{4} (N_1 - n_1)^2
\end{aligned}$$

and the similar inequalities for the mapping  $g$ . ■

**Remark 2.** If  $a = c$ ,  $b = d$ , then  $m_1 = m_2 =: m$ ,  $M_1 = M_2 =: M$ ,  $M(f; a, b) = M(f; c, d)$  and  $T(f, g; a, b, c, d) = 2T(f, g; a, b)$ , and then from (2.10), we recapture the Grüss inequality (1.2).

**Corollary 2.** Assume that  $f$  and  $g$  are absolutely continuous on  $\bar{I}$ . In addition, assume that  $f', g' \in L_\infty(\dot{I})$  and  $[a, b], [c, d] \subseteq \bar{I}$  ( $\bar{I}$  is the closure of  $I$ ). Then we have

the inequality

$$\begin{aligned}
 (2.11) \quad & |T(f, g; a, b, c, d)| \\
 & \leq \left[ \frac{1}{12} \|f'\|_{\infty, [a, b]}^2 (b-a)^2 + \frac{1}{12} \|f'\|_{\infty, [c, d]}^2 (d-c)^2 \right. \\
 & \quad \left. + (M(f; a, b) - M(f; c, d))^2 \right]^{\frac{1}{2}} \\
 & \times \left[ \frac{1}{12} \|g'\|_{\infty, [a, b]}^2 (b-a)^2 + \frac{1}{12} \|g'\|_{\infty, [c, d]}^2 (d-c)^2 \right. \\
 & \quad \left. + (M(g; a, b) - M(g; c, d))^2 \right]^{\frac{1}{2}},
 \end{aligned}$$

where

$$\|f'\|_{\infty, [a, b]} := \operatorname{ess\,sup}_{t \in [a, b]} |f'(t)|.$$

The proof follows by (2.3) and utilising the fact that, by the Chebychev inequality (1.3),

$$T(f; a, b) \leq \frac{1}{12} \|f'\|_{\infty}^2 (b-a)^2.$$

**Remark 3.** Similarly, if  $a = c$ ,  $b = d$ , then by (2.11) we deduce Chebychev's inequality (1.3).

Finally, by the inequality due to Lupaş (1.4), we may state the following corollary.

**Corollary 3.** Assume that  $f$  and  $g$  are absolutely continuous on  $\bar{I}$ ,  $f', g' \in L_2(\bar{I})$  and  $[a, b], [c, d] \subset \bar{I}$ . Then we have the inequality

$$\begin{aligned}
 (2.12) \quad & |T(f, g; a, b, c, d)| \\
 & \leq \left[ \frac{1}{\pi^2} \|f'\|_{2, [a, b]}^2 (b-a)^2 + \frac{1}{\pi^2} \|f'\|_{2, [c, d]}^2 (d-c)^2 \right. \\
 & \quad \left. + (M(f; a, b) - M(f; c, d))^2 \right]^{\frac{1}{2}} \\
 & \times \left[ \frac{1}{\pi^2} \|g'\|_{2, [a, b]}^2 (b-a)^2 + \frac{1}{\pi^2} \|g'\|_{2, [c, d]}^2 (d-c)^2 \right. \\
 & \quad \left. + (M(g; a, b) - M(g; c, d))^2 \right]^{\frac{1}{2}},
 \end{aligned}$$

where

$$\|f'\|_{2, [a, b]} := \left( \int_a^b |f'(t)|^2 dt \right)^{\frac{1}{2}}.$$

**Remark 4.** If  $a = c$ ,  $b = d$ , then by (2.12), we recapture the inequality (1.4) due to Lupaş.

We define another generalised Chebychev functional involving the mean of the product of two functions, and the product of the means of each of the functions, where one is over a different interval, by

$$(2.13) \quad \mathfrak{T}(f, g; a, b, c, d) := M(fg; a, b) - M(f; a, b) M(g; c, d),$$

where  $M(\cdot, \cdot, \cdot)$  is as defined in (2.2). It may be noticed from (2.13) and (1.1) that  $\mathfrak{T}(f, g; a, b; a, b) = T(f, g; a, b)$ .

It may be noticed that (2.13) is related to (1.1) by the identity

$$(2.14) \quad T(f, g; a, b, c, d) = \mathfrak{T}(f, g; a, b, c, d) + \mathfrak{T}(g, f; c, d, a, b).$$

**Theorem 2.** *Let  $f, g : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be measurable on  $I$  and the intervals  $[a, b], [c, d] \subset I$ . In addition, let  $m_1 \leq f \leq M_1$  and  $n_1 \leq g \leq N_1$  a.e. on  $[a, b]$  with  $n_2 \leq g \leq N_2$  a.e. on  $[c, d]$ . Then the following inequalities hold*

$$(2.15) \quad \begin{aligned} & |\mathfrak{T}(f, g; a, b, c, d)| \\ & \leq [T(f; a, b) + M^2(f; a, b)]^{\frac{1}{2}} \\ & \quad \times [T(g; a, b) + T(g; c, d) + (M(g; a, b) - M(g; c, d))^2]^{\frac{1}{2}} \\ & \leq \left[ \left( \frac{M_1 - m_1}{2} \right)^2 + M^2(f; a, b) \right]^{\frac{1}{2}} \\ & \quad \times \left[ \left( \frac{N_1 - n_1}{2} \right)^2 + \left( \frac{N_2 - n_2}{2} \right)^2 + (M(g; a, b) - M(g; c, d))^2 \right]^{\frac{1}{2}}, \end{aligned}$$

where  $T(f; a, b)$  is as given by (2.4) and  $M(f; a, b)$  by (2.2).

*Proof.* The proof follows closely that of Theorem 1 and Corollary 1. From (2.14) and (2.5), or directly from (2.13), it may be shown that

$$\mathfrak{T}(f, g; a, b, c, d) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x)(g(x) - g(y)) dy dx,$$

and thus, using the Cauchy-Buniakowski-Schwartz inequality for double integrals gives

$$\begin{aligned} & |\mathfrak{T}(f, g; a, b, c, d)|^2 \\ & = \left[ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x)(g(x) - g(y)) dy dx \right]^2 \\ & \leq \left( \frac{1}{b-a} \int_a^b f^2(x) dx \right) \left( \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d (g(x) - g(y))^2 dy dx \right) \\ & = M(f^2; a, b) T(g, g; a, b, c, d), \end{aligned}$$

which, upon using (2.4) and (2.7) produces the first inequality in (2.15).

Using the Grüss inequality (1.2), identity (2.4) together with the bounds on  $f$  and  $g$  over the respective intervals produces the second inequality, and the theorem is proved. ■

**Remark 5.** *If the Chebychev and Lupas bounds were used with the first inequality in (2.15), then similar results to those obtained in Corollaries 2 and 3 for  $T(g, f; a, b, c, d)$  can be obtained for  $\mathfrak{T}(f, g; a, b, c, d)$  as defined in (2.13).*

## 3. APPLICATIONS IN NUMERICAL ANALYSIS

If in (2.3) we assume that  $c = z - \varepsilon$ ,  $d = z + \varepsilon$  where  $z \in (a, b)$  and  $\varepsilon > 0$  is chosen accordingly such that  $c, d \in (a, b)$ , then, letting  $\varepsilon \rightarrow 0+$  we get from (2.3) and, using (2.1), (2.2) and (2.4), the following inequality

$$\begin{aligned}
(3.1) \quad & \left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - g(z) \frac{1}{b-a} \int_a^b f(x) dx \right. \\
& \left. - f(z) \frac{1}{b-a} \int_a^b g(x) dx + f(z) g(z) \right| \\
& \leq \left[ \frac{1}{b-a} \int_a^b f^2(x) dx - \left( \frac{1}{b-a} \int_a^b f(x) dx \right)^2 \right. \\
& \quad \left. + \left( \frac{1}{b-a} \int_a^b f(x) dx - f(z) \right)^2 \right]^{\frac{1}{2}} \\
& \quad \times \left[ \frac{1}{b-a} \int_a^b g^2(x) dx - \left( \frac{1}{b-a} \int_a^b g(x) dx \right)^2 \right. \\
& \quad \left. + \left( \frac{1}{b-a} \int_a^b g(x) dx - g(z) \right)^2 \right]^{\frac{1}{2}} \\
& : = K.
\end{aligned}$$

If, for example, we assume that  $f$  and  $g$  satisfy the assumptions of Corollary 2, then we have:

$$\begin{aligned}
(3.2) \quad K & \leq \left[ \frac{1}{12} \|f'\|_{\infty, [a, b]}^2 (b-a)^2 + \left( \frac{1}{b-a} \int_a^b f(x) dx - f(z) \right)^2 \right]^{\frac{1}{2}} \\
& \quad \times \left[ \frac{1}{12} \|g'\|_{\infty, [a, b]}^2 (b-a)^2 + \left( \frac{1}{b-a} \int_a^b g(x) dx - g(z) \right)^2 \right]^{\frac{1}{2}}.
\end{aligned}$$

Similar bounds for  $K$  can be obtained if we use Corollary 1 or Corollary 3 as indeed for  $\mathfrak{F}(f, g; a, b, c, d)$  of Theorem 2. For a given division  $\delta : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$  of  $[a, b]$  having the intermediate points  $\xi_i \in [x_i, x_{i+1}]$ ,  $i = 0, \dots, n-1$ , we can construct the formula

$$\begin{aligned}
(3.3) \quad & A(f, g, \delta, \xi) \\
& : = \sum_{i=0}^{n-1} g(\xi_i) \int_{x_i}^{x_{i+1}} f(t) dt + \sum_{i=0}^{n-1} f(\xi_i) \int_{x_i}^{x_{i+1}} g(t) dt - \sum_{i=0}^{n-1} h_i f(\xi_i) g(\xi_i),
\end{aligned}$$

where  $h_i := x_{i+1} - x_i$  ( $i = 0, \dots, n-1$ ).

We may state the following result which gives an approximation of the integral of the product, namely,  $\int_a^b f(x) g(x) dx$  in terms of the sum  $A(f, g, \delta, \xi)$ .

**Theorem 3.** Assume that the mapping  $f$  and  $g$  satisfy the assumptions in Corollary 2. Then we have, for  $\delta$  and  $\xi$  as above:

$$(3.4) \quad \int_a^b f(x)g(x)dx = A(f, g, \delta, \xi) + R(f, g, \delta, \xi)$$

where the remainder  $R(f, g, \delta, \xi)$  satisfies the bound

$$(3.5) \quad \begin{aligned} & |R(f, g, \delta, \xi)| \\ & \leq \sum_{i=0}^{n-1} h_i \left[ \frac{1}{12} \|f'\|_{\infty, [a, b]}^2 h_i^2 + \left( \frac{1}{h_i} \int_{x_i}^{x_{i+1}} f(x) dx - f(\xi_i) \right)^2 \right]^{\frac{1}{2}} \\ & \quad \times \left[ \frac{1}{12} \|g'\|_{\infty, [a, b]}^2 h_i^2 + \left( \frac{1}{h_i} \int_{x_i}^{x_{i+1}} g(x) dx - g(\xi_i) \right)^2 \right]^{\frac{1}{2}} \\ & \leq \left[ \frac{1}{12} \|f'\|_{\infty, [a, b]}^2 \sum_{i=0}^{n-1} h_i^3 + \sum_{i=0}^{n-1} h_i \left( \frac{1}{h_i} \int_{x_i}^{x_{i+1}} f(x) dx - f(\xi_i) \right)^2 \right]^{\frac{1}{2}} \\ & \quad \times \left[ \frac{1}{12} \|g'\|_{\infty, [a, b]}^2 \sum_{i=0}^{n-1} h_i^3 + \sum_{i=0}^{n-1} h_i \left( \frac{1}{h_i} \int_{x_i}^{x_{i+1}} g(x) dx - g(\xi_i) \right)^2 \right]^{\frac{1}{2}} \\ & \leq \left[ \frac{1}{12} \|f'\|_{\infty, [a, b]}^2 \sum_{i=0}^{n-1} h_i^3 + \|f'\|_{\infty, [a, b]} \sum_{i=0}^{n-1} \left( \frac{1}{4} + \frac{\left( \xi_i - \frac{x_i + x_{i+1}}{2} \right)^2}{h_i^2} \right)^2 h_i^3 \right]^{\frac{1}{2}} \\ & \quad \times \left[ \frac{1}{12} \|g'\|_{\infty, [a, b]}^2 \sum_{i=0}^{n-1} h_i^3 + \|g'\|_{\infty, [a, b]} \sum_{i=0}^{n-1} \left( \frac{1}{4} + \frac{\left( \xi_i - \frac{x_i + x_{i+1}}{2} \right)^2}{h_i^2} \right)^2 h_i^3 \right]^{\frac{1}{2}} \\ & \leq \frac{1}{3} \|f'\|_{\infty, [a, b]} \|g'\|_{\infty, [a, b]} \sum_{i=0}^{n-1} h_i^3 \rightarrow 0 \text{ as } \nu(\delta) := \max_{i=0, n-1} h_i \rightarrow 0. \end{aligned}$$

*Proof.* The first inequality in (3.4) follows by applying (3.1) and (3.2) on the intervals  $[x_i, x_{i+1}]$  with  $\xi_i \in [x_i, x_{i+1}]$  and summing the resulting inequalities.

The second inequality follows by the Cauchy-Schwartz inequality with the positive weight  $h_i$  ( $i = 0, \dots, n-1$ ).

The third inequality results from using Ostrowski's inequality (see [2, p. 468])

$$\left| \frac{1}{h_i} \int_{x_i}^{x_{i+1}} f(x) dx - f(\xi_i) \right| \leq \left[ \frac{1}{4} + \frac{\left( \xi_i - \frac{x_i + x_{i+1}}{2} \right)^2}{h_i^2} \right] \|f'\|_{\infty, [a, b]} h_i$$

for all  $i = 0, \dots, n-1$ , and the last inequality is obtained on using the fact that

$$\left( \xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \leq \frac{1}{4} h_i^2, \quad i = 0, \dots, n-1.$$

■

## REFERENCES

- [1] J. PEČARIĆ, F. PROCHAN and Y. TONG, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, San Diego, 1992.
- [2] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.
- [3] S.S. DRAGOMIR, Grüss inequality in inner product spaces, *The Australian Math Soc. Gazette*, **26** (1999), No. 2, 66-70.
- [4] S.S. DRAGOMIR, A generalization of Grüss' inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, **237** (1999), 74-82.
- [5] S.S. DRAGOMIR and I. FEDOTOV, An inequality of Grüss' type for Riemann-Stieltjes integral and applications for special means, *Tamkang J. of Math.*, **29**(4)(1998), 286-292.
- [6] S.S. DRAGOMIR, A Grüss type integral inequality for mappings of  $r$ -Hölder's type and applications for trapezoid formula, *Tamkang J. of Math.*, (in press).
- [7] S.S. DRAGOMIR, Some integral inequalities of Grüss type, *Indian J. of Pure and Appl. Math.*, (accepted).
- [8] S.S. DRAGOMIR and G.L. BOOTH, On a Grüss-Lupaş type inequality and its applications for the estimation of  $p$ -moments of guessing mappings, *Mathematical Communications*, (accepted).
- [9] G. Grüss, Über das Maximum des absoluten Betrages von  $\frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx$ , *Math. Z.*, **39**(1935), 215-226.
- [10] A.M. FINK, A treatise on Grüss' inequality, *submitted*.

SCHOOL OF COMMUNICATIONS AND INFORMATICS, VICTORIA UNIVERSITY OF TECHNOLOGY, PO BOX 14428, MELBOURNE CITY MC, VICTORIA, 8001, AUSTRALIA.

*E-mail address:* [pc@matilda.vu.edu.au](mailto:pc@matilda.vu.edu.au)

*URL:* <http://sci.vu.edu.au/staff/peterc.html>

*E-mail address:* [sever.dragomir@vu.edu.au](mailto:sever.dragomir@vu.edu.au)

*URL:* <http://rgmia.vu.edu.au/SSDragomirWeb.html>