# ON AN INEQUALITY AND A MEAN VALUE

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**Abstract. Using the papers [1] and [2] the inequality (1) is given and the author indicates the application of the given inequality on two particular cases. In the context of one of these particular cases we discovered a mean $M(a,b)$ on the segment $[a,b]$ labeled as $H(a,b) < M(a,b) < G(a,b)$, together with a geometric interpretation which is given, according to paper [3], by the relation $M = H(A,H)$.**

**$1^0$.** The following theorem is true.

**Theorem.** *If for the function $f(x)$ on interval $(a,b)$ $f''(x) \underset{(>)}{<} 0$ is true and $f(x)$ has the corresponding one-sided final derivates $f'(a)$ and $f'(b)$ ( $f'(b) \neq f'(a)$ ), then*

$$\int_a^b f(x)\,dx \underset{(>)}{<} \frac{f(a)+f(b)}{2}(b-a) - \frac{1}{8}\left[f'(b)-f'(a)\right](b-a)^2, \tag{1}$$

*respectively.*

**Proof.** According to theorem $3'$ in [1] pg. 161, the following is true

$$p = \frac{X-a}{b-a}, \qquad q = \frac{b-X}{b-a}; \tag{2}$$

$X$ − abscissa of the intersection point of tangents in the points $A[a, f(a)]$ and $B[b, f(b)]$.

Let consider the first case given in figure 1 where the tangents in border points $A$ and $B$

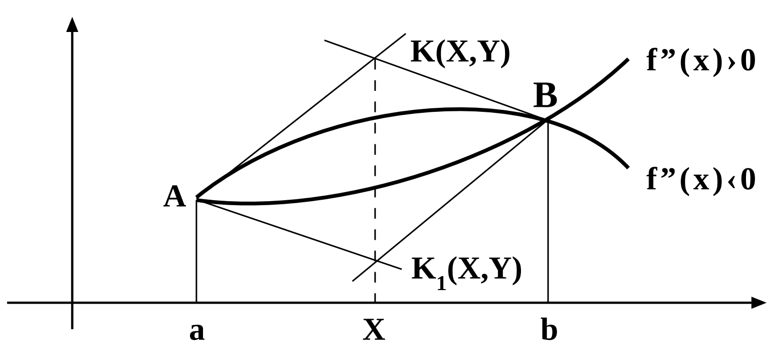


**Figure 1.**

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of the graphic of function $f(x)$ intersect above the $x$-axis. In the case of $f^{''}(x) < 0$, [2] is true, pg. 69:

$$P_{\triangle}(AKB) = \frac{1}{2}|\overrightarrow{BK} \cdot \overrightarrow{KA}^{'}|, \tag{3}$$

where $\overrightarrow{KA}^{'}$ is directly ortogonal vector to the vector $\overrightarrow{KA}$. Next,

$$\overrightarrow{KA}^{'} = (-\overrightarrow{KA})' = -\big[(X-a)\vec{i}+(Y-f(a))\vec{j}\,\big]^{'} = -\big[p(b-a)\vec{i}+pf^{'}(a)(b-a)\vec{j}\,\big]^{'} = -p(b-a)(-f^{'}(a)\vec{i}+\vec{j})$$
$$\wedge \qquad \overrightarrow{BK} = (X-b)\vec{i}+(Y-f(b))\vec{j} = -q(b-a)\vec{i}-q(b-a)f^{'}(b)\vec{j} = -q(b-a)(\vec{i}-f^{'}(b)\vec{j})$$
$$\Longrightarrow \qquad \overrightarrow{BK}\cdot\overrightarrow{KA}^{'} = pq(b-a)^2(f^{'}(b)-f^{'}(a)).$$

Finally from

$$P_{\triangle}(AKB) = \frac{1}{2}pq(b-a)^2|f^{'}(b)-f^{'}(a)|, \tag{4}$$

using the implication

$$f^{''}(x) < 0 \Longrightarrow f^{'}(b) < f^{'}(a) \tag{5}$$

it follows that

$$P_{\triangle}(AKB) = -\frac{1}{2}pq(b-a)^2(f^{'}(b)-f^{'}(a)). \tag{6}$$

Thus

$$\int\limits_a^b f(x)\,dx < \frac{f(a)+f(b)}{2}(b-a)+P_{\triangle}(AKB) = \frac{f(a)+f(b)}{2}(b-a)-\frac{1}{2}pq\big[f^{'}(b)-f^{'}(a)\big](b-a)^2. \tag{7}$$

From $p+q=1 \Longrightarrow pq = p(1-p) = p-p^2$. Function $f(p) = p-p^2$ has the maximum in the point $p = \frac{1}{2}$ and $\varphi_{\max}(p) = \varphi(\frac{1}{2}) = \frac{1}{4}$, so that finally

$$\int\limits_a^b f(x)\,dx < \frac{f(a)+f(b)}{2}(b-a) - \frac{1}{8}\big[f^{'}(b)-f^{'}(a)\big](b-a)^2, \quad \text{for} \quad f^{''}(x) < 0. \tag{8}$$

In case of $f^{''}(x) > 0$ there is

$$P_{\triangle}(AK_1B) = \frac{1}{2}|\overrightarrow{AK_1}\cdot\overrightarrow{K_1B}^{'}| = \frac{1}{2}|\overrightarrow{AK_1}\cdot\overrightarrow{BK_1}^{'}|,$$

where

$$\overrightarrow{AK_1} = (X-a)\vec{i} + \big[Y-f(a)\big]\vec{j} = p(b-a)\vec{i}+pf^{'}(a)(b-a)\vec{j} = p(b-a)P\big[\vec{i}+f^{'}(a)\vec{j}\big],$$
$$\overrightarrow{BK_1} = -q(b-a)\big[\vec{i}+f^{'}(b)\vec{j}\big], \qquad \overrightarrow{BK_1}^{'} = q(b-a)\big[f^{'}(b)\vec{i}-\vec{j}\big].$$

It follows that

$$P_{\triangle}(AK_1B) = \frac{1}{2}pq(b-a)^2|f^{'}(b)-f^{'}(a)|.$$

Next, using the implication

$$f^{''}(x) > 0 \Longrightarrow f^{'}(b) > f^{'}(a), \tag{9}$$

it follows that

$$P_{\triangle}(AK_1B)=\frac{1}{2}pq\big[f^{'}(b)-f^{'}(a)\big](b-a)^2. \tag{10}$$

Thus

$$\int_a^b f(x)\,dx>\frac{f(a)+f(b)}{2}(b-a)-P_{\triangle}(AK_1B)=\frac{f(a)+f(b)}{2}(b-a)-\frac{1}{2}pq\big[f^{'}(b)-f^{'}(a)\big](b-a)^2,$$

so that finally

$$\int_a^b f(x)\,dx>\frac{f(a)+f(b)}{2}(b-a)-\frac{1}{8}\big[f^{'}(b)-f^{'}(a)\big](b-a)^2,\quad \text{for}\quad f^{''}(x)>0. \tag{11}$$

In the second case, the results (8) and (9) are true for the function $y=f(x)+C$, where the constant $C>0$ is chosen in the way that the figure 1, from the previous consideration, is attained. Namely then, according to the above mentioned

$$\int_a^b f(x)\,dx\;+\;C(b-a)\underset{(>)}{<}\frac{f(a)+f(b)}{2}(b-a)+C(b-a)-\frac{1}{8}\big[f^{'}(b)-f^{'}(a)\big](b-a)^2, \tag{12}$$

the conclusions follow. By this the theorem is completely proven.

**$2^0$.** At this point we shall apply the theorem to the two particular cases.

**1)** In the case of the function

$$f(x)=\ln x,\quad x>0, \tag{13}$$

there is

$$f^{'}(x)=\frac{1}{x},\; f^{''}(x)=-\frac{1}{x^2}\;<0,$$

so by the applying the theorem on the segment $[a,b]$ $(b>a>0)$ it is true that

$$\int_a^b \ln x\,dx<\frac{\ln a+\ln b}{2}(b-a)-\frac{1}{8}\Big(\frac{1}{b}-\frac{1}{a}\Big)(b-a)^2. \tag{14}$$

For $a>b>0$ we would have in the previous inequality the opposite sign.

The calculations induce

$$(\ln b-\ln a)\frac{a+b}{2}<\frac{a^2+6ab+b^2}{8ab}(b-a)$$

in fact to the inequality

$$\frac{\ln b-\ln a}{b-a}\,\frac{a+b}{2}<\frac{a^2+6ab+b^2}{8ab},$$

which is true for $b<a$. Finally, the following relation is true

$$\frac{\ln b-\ln a}{b-a}<\frac{a^2+6ab+b^2}{4ab(a+b)},\quad b>a>0\ \text{ or }\ a>b>0. \tag{15}$$

According to (15) the obtained bounds of the fraction $\dfrac{\ln b - \ln a}{b-a}$ in the left and right neighbourhood of the point a are of *an invariant form.*

**$\alpha$)** For $a = 1$ and $b = 1 + x$, $x > 0$, from relation (15) it follows that

$$\ln(1+x) < x - \frac{3x^3 + 4x^2}{4(1+x)(2+x)}, \quad x > 0; \tag{16}$$

while for $a = 1$ and $b = 1 + x$, $-1 < x < 0$, from the mentioned relation it follows that

$$\ln(1+x) > x - \frac{3x^3 + 4x^2}{4(1+x)(2+x)}, \quad -1 < x < 0. \tag{17}$$

*Remark.* By the relations (16) and (17) for $f(x) = \ln(1+x)$, in the domain of rational functions, we got more precise bounds than the bounds in (35) and (35′) respectively from the papers [1], pg. 173, that is, 175. Let notice at this point that the relations (35) and (35′) in [1] are by technical error labeled as (35) and (35′). Namely, with (35) and (35′) we should have labeled the relations two rows before these relations respectively. In the context of that remark, the inequality (35”) in [1], pg. 175, becomes

$$R_2\big[\ln(1+x)\big] < \frac{x^2}{1 \pm |x|}, \quad 0 < |x| < 1; \tag{18}$$

with a remark that for $x \in (-1, 0)$ the following is true

$$R_2\big[\ln(1+x)\big] < \frac{x^2}{2(1+x)}. \tag{19}$$

**$\beta$)** From relation (15) the following relation is true

$$L(a,b) = \frac{b-a}{\ln b - \ln a} > \frac{4ab(a+b)}{a^2 + 6ab + b^2} = M(a,b), \tag{20}$$

where $M(a,b)$ is also a mean of the numbers $a$ and $b$ $\big(M(a,a) = a$ and $a < M(a,b) < b\big)$. It is not difficult to verify the relations

$$H(a,b) < M(a,b) < G(a,b). \tag{21}$$

The relation $M(a,b) < G(a,b)$ is got from the accuracy of the relation

$$G(G, G') < A(G, G'), \tag{22}$$

where $G' = KG -$ complementary mean of the geometric mean [3].

For the geometric construction of the mean $M(a,b)$ through the classic means, we determine the previously reciprocal mean for $M(a,b)$ in relation to number $ab$ [3]; let label it with $X$, by notation $RM = X$, it is true that

$$M \cdot X = ab. \tag{23}$$

It follows that

$$X = \frac{a^2 + 6ab + b^2}{4(a+b)} = \frac{(a+b)^2}{4(a+b)} + \frac{ab}{a+b} = \frac{1}{2}\Big(\frac{a+b}{2} + \frac{2ab}{a+b}\Big),$$

ie

$$X(a,b) = \frac{A(a,b) + H(a,b)}{2}. \tag{24}$$

Thus, according to [3], we finally get

$$M = RX = R\Big(\frac{A+H}{2}\Big) = H(A,H). \tag{25}$$

According to the figure 1 from the paper [3] we construct the means $A = A(a,b)$, $G = G(a,b)$ and $H = H(a,b)$; and then the mean $M$, according to the figure 2.

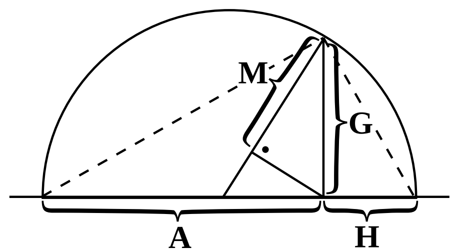


**Figure 2.**

**2)** In the case of the function

$$f(x) = \sin x, \quad 0 < x < \pi, \tag{26}$$

there is

$$f^{'}(x) = \cos x, \;\; f^{''}(x) = -\sin x,$$

so, by the applying the theorem on the segment $[0,x]$, $x < \pi$, we have that

$$\int_0^x \sin x \, dx < \frac{x \sin x}{2} - \frac{x^2}{8}(\cos x - 1), \tag{27}$$

which induce the relation

$$\tan\frac{x}{2}\left(2 - \frac{x^2}{4}\right) < x.$$

Finally

$$\tan\frac{x}{2} < \frac{4x}{8 - x^2}, \quad \text{for} \quad 0 < x < 2\sqrt{2}. \tag{28}$$

For $-\pi < x < 0$ we would previously have the opposite sign, so by substitution of $\dfrac{x}{2} = t$ we got the following inequality

$$\tan|t| < \frac{2|t|}{2 - t^2}, \quad 0 < |t| < \sqrt{2}. \tag{29}$$

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